Empirical likelihood control charts for the process mean

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Quality

- ***Quality*** is an important concept in industry that involves several factors and affects considerably the choices of a customer.

- ***Statistical quality control*** is one of the main tools for monitoring and improving quality of goods.

- One of the main areas of application of the statistical quality control is that of the ***process control***.

- The process control is usually based on standard statistical test procedures applied to ***quality control charts***.

Control chart for m

- The standard method to ***control the mean*** proceed as follows:
  - For any time interval we ***draw a sample*** of n pieces and for this sample we compute the mean of the variable of interest, X, denoted by $\overline{X}$.
  - The sample means are represented on a plot together with the ***expected mean*** (m_0), the ***lower control limit*** (LCL) and ***upper control limit*** (UCL)

$$\text{LCL} = m_0 - z_{a/2}\sqrt{s^2/n}, \qquad \text{UCL} = m_0 + z_{a/2}\sqrt{s^2/n},$$

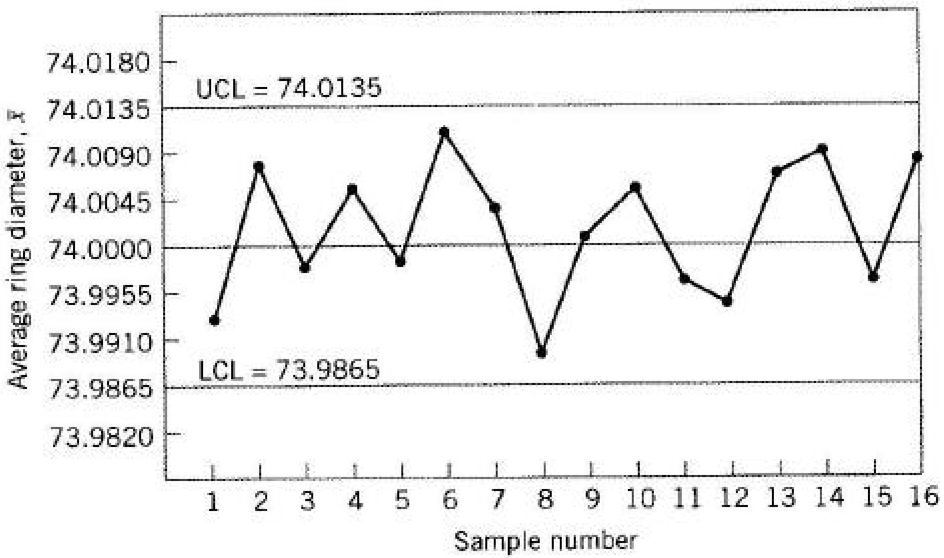
    where a is the probability of the type I error and $z_{a/2}$ is the $100(1-a/2)$th percentile of the standard Normal distribution (usually we follow the $3s$ rule and so we substitute $z_{a/2}$ with 3).
  - If a point is out of the control limits, we have to ***stop the process***.

Example

- We expect that length of a certain product has average $m_0 = 74$mm.

- The process is followed by using samples of size $n = 5$ drawn each day. So, with the $3s$ rule and assuming $s^2 = 0{,}0001$, the control limits are

$$\text{LCL} = 74 - 3\sqrt{0{,}0001/5} = 73{,}9865 \qquad \text{UCL} = 74 + 3\sqrt{0{,}0001/5} = 74{,}0135$$



Power of the control chart

- The power of a control chart is usually measured through the ***average run length*** (ARL), equal to the average number of samples that we have to observe before the control charts shows that the process is out of control when it is true;

$$\text{ARL} = \frac{1}{1-b},$$

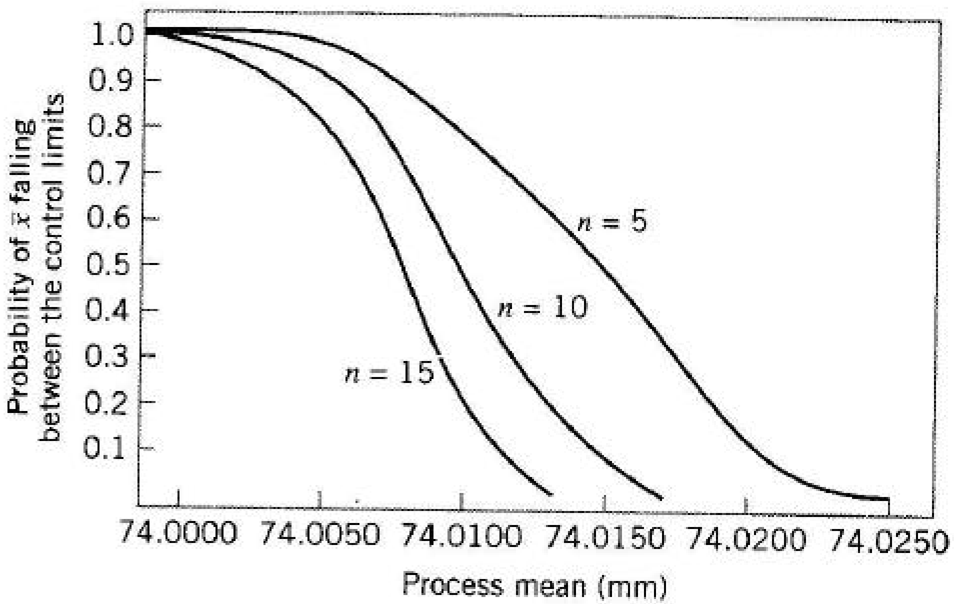
where $b = P(\text{not rejct } m_0 \mid m_0 \text{ is not true})$ is the probability of the type II error.

- The ARL ***increases*** when b increses.

- When the ARL is ***small***, the control chart is powerful, in the sense that it requires a few samples to show that the process is out of control.

- Usually the power of the chart ***increases*** with the sample size (n) and with the frequency with that samples are drawn; so, it may be convenient to increase the sample size, but this is expensive and time consuming.

- For the previous example, we have the following ***operating characteristic curve*** that show how the power of the chart varies with n when the true mean (m) is different from the expected one (m_0).



- As we can see, the probability of the type II error (b) ***decreases*** when the sample size (n) increases.

- An alternative measure is the ***average time to signal*** (ATS) that is the average time, instead of the average number of samples, necessary to the control chart to show that the process is out of control;

$$\text{ATS} = t \cdot \text{ARL},$$

where t is the time between sample draws.

- By using ARL and/or ATS we can ***compare*** two control charts based on two different sampling schemes.

Warning area

- The ***probability of the type I error*** (a) is usually fixed at a level close to 0 so that the chance of stopping the process when it is in control is very low.

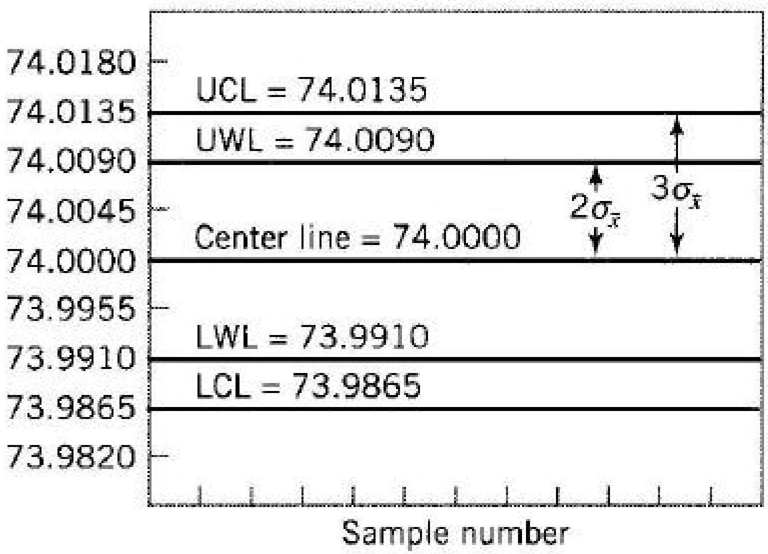
- However, when a is very small, the control chart is ***not powerful enough***, in the sense it is not easy that the chart shows that the process is out of control this is necessary.

- To increase the power of a chart we usually make use the so-called ***warning limits*** that correspond to higher levels of a. These limits are often chosen with the $2s$ rule:

$$\mathrm{LWL} = m_0 - 2\sqrt{s^2 / n} \quad \text{and} \quad \mathrm{UWL} = m_0 + 2\sqrt{s^2 / n}.$$

- For the previous example we have

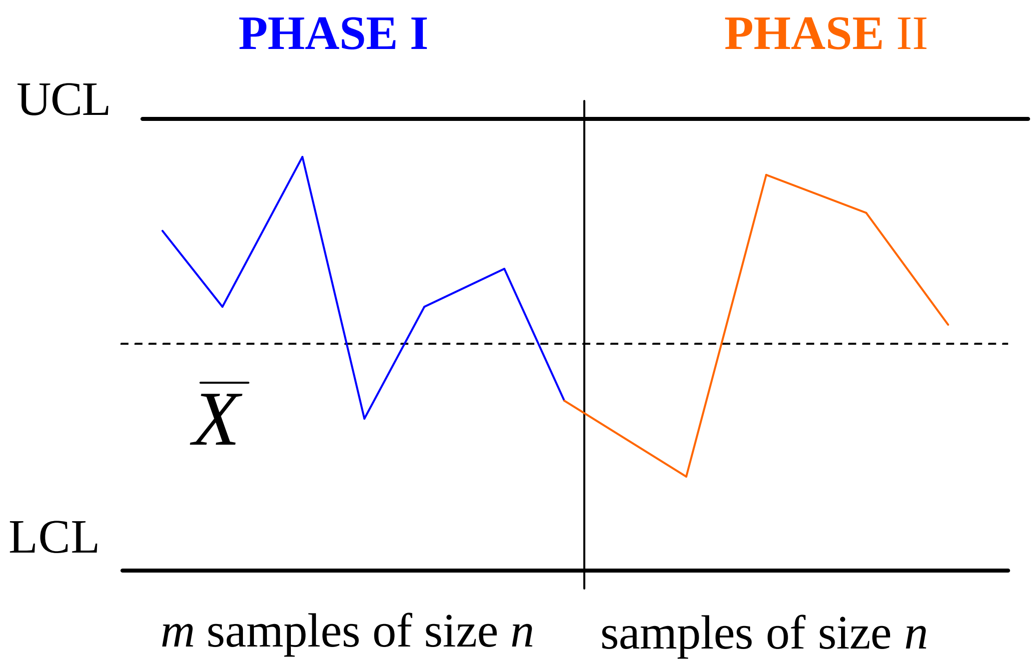
$$LWL = 74 - 2\sqrt{0.0001/5} = 73{,}9910 \qquad UWL = 74 + 2\sqrt{0.0001/5} = 74{,}0090$$



- If the process is in control, points in the ***warning area*** (between LWL and LCL and between UCL and UWL) are uncommon and so, with the aim of increasing the power of the chart, we stop the process if we have many points in this area.

When s^2 is not known

- If we have no idea of the value of the ***population variance*** (s^2), we can replace this parameter with an estimate based on m samples drawn when the process was surely under control, the so-called ***phase I***.



- The ***variance estimate*** at issue is

$$S^2 = \frac{1}{mn-1}\sum_{i=1}^{m}\sum_{j=1}^{n}(X_{ij}-\overline{\overline{X}})^2 ,$$

where $\overline{\overline{X}} = \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} X_{ij}$ is the mean of all the observations in phase I.

- Consequently, the ***control limits*** become

$$\text{LCL} = m_0 - 3\sqrt{S^2/n} \quad \text{and} \quad \text{UCL} = m_0 + 3\sqrt{S^2/n}$$

while the ***warning limits*** will be

$$\text{LWL} = m_0 - 2\sqrt{S^2/n} \quad \text{and} \quad \text{UWL} = m_0 + 2\sqrt{S^2/n}$$

When m_0 *and* s^2 *are not known*

- When we also do not know the ***population mean*** (m_0) when we process is in control, we can still use a suitable number of m samples drawn in phase I and substitute m_0 with the estimate $\overline{\overline{X}}$.

- Consequently, the ***control limits*** become ($3s$ rule)

$$\text{LCL} = \overline{\overline{X}} - 3\sqrt{S^2/n} \quad \text{and} \quad \text{UCL} = \overline{\overline{X}} + 3\sqrt{S^2/n}$$

while the ***warning limits*** will be

$$\text{LWL} = \overline{\overline{X}} - 2\sqrt{S^2/n} \quad \text{and} \quad \text{UWL} = \overline{\overline{X}} - 2\sqrt{\overline{S}^2/n}$$

Possible Problems

- All the previous procedures are based on the ***assumption of normality*** of the variable of interest, X. In many situations, however, the actual distribution may significantly depart from normality.

- The ***consequences*** are usually that:
    - the actual level a is different from the nominal one;
    - the power to detect process shifts is normally lower than expected.

- Some ***control charts*** have been proposed to deal with these situations. One of the most-well known is the bootstrap control chart that, however, does not work so well as expected.

- We propose a control charts based on the ***empirical likelihood*** (EL).

Empirical Likelihood

- ***Empirical Likelihood*** (EL) is a nonparametric method to test hypotheses on a population parameters (Owen, 1988).

- The approach is based on the a sort of ***nonparametric likelihood*** that, when the parameter of interest is the population mean (m), may be expressed as

$$L(m) = \max_{\mathbf{p}:\sum_i p_i X_i = m} \prod_{i=1}^{n} p_i \,,$$

  where $\mathbf{p} = (p_1, \ldots, p_n)'$ is a vector of probabilities summing up to 1.

- The function at issue is in practice the ***profile likelihood*** (maximum of the likelihood with respect to $\mathbf{p}$) of a multinomial model that places a weight p_i on any observation X_i.

- An interesting property is that $L(m)$ reaches its ***maximum*** when m is equal to the sample mean $\bar{X}$. This happens when all the weights are equal,

$$p_i = \frac{1}{n}, \qquad i = 1, \ldots, n,$$

and so $L(\bar{X}) = n^{-n}$.

- This allows to consider the ***non parametric likelihood ratio***

$$R(m) = \frac{L(m)}{L(\bar{X})} = \max_{\mathbf{p}:\sum_i p_i X_i = m} \prod_{i=1}^{n} np_i$$

to the hypothesis $H_0 : m = m_0$ versus $H_1 : m \neq m_0$, without assuming any specific distribution for X.

- The ***likelihood ratio*** $R(m)$ is always between 0 and 1 and in general the larger the value of $R(m)$, the higher the evidence provided by the data in favor of m as the true value of the population mean.

- So, we ***reject*** the hypothesis $H_0 : m = m_0$ in favor of $H_1 : m \neq m_0$ when $R(m_0)$ is smaller than a suitable threshold, r_0.

- The most used criterion to define r_0 is based on the ***asymptotic theory***. It is in fact possible to prove that under H_0

$$l(m_0) = -2\log(R(m_0)) \xrightarrow{D} c_1^2.$$

This is an extension of the Wilk (1938) theorem which is well-known in parametric inference.

- So we can ***formulate the test*** as

$$\text{reject } H_0 \text{ if } l(m_0) \geq c_a$$

where c_a is the $(1-a)100$-th percentile of the c_1^2 distribution.

- Unfortunately, the ***convergence*** of the true distribution of $l(m_0)$ to the c_1^2 distribution is quite slow and so some methods have been developed to find more precise thresholds for $l(m_0)$. The most used methods are based on:

    - Bartlett correction

    - Bootstrap method

Empirical Likelihood control charts

- When we apply EL to control charts, we cannot use a ***calibration*** based on the asymptotic c_1^2 distribution since the sample size is usually very small.

- However, can take advantage of the ***presence of phase I*** observations to calibrate as follows:
  - draw with replacement T samples of size n from the set of all the mn observations in phase I and compute $l(m_0)$ for any of them obtaining

$$l_1(m_0),\ldots,l_T(m_0)$$

  - the upper control limit for the chart (UCL) is equal to the $100(1-a)$-th percentile of the set of test statistics above.

- The percentile may be computed by arranging in ***increasing order*** the test statistics $l(m_0)$'s and then taking the $T(1-a)$ largest value.

- The ***ordered set*** will be denoted by

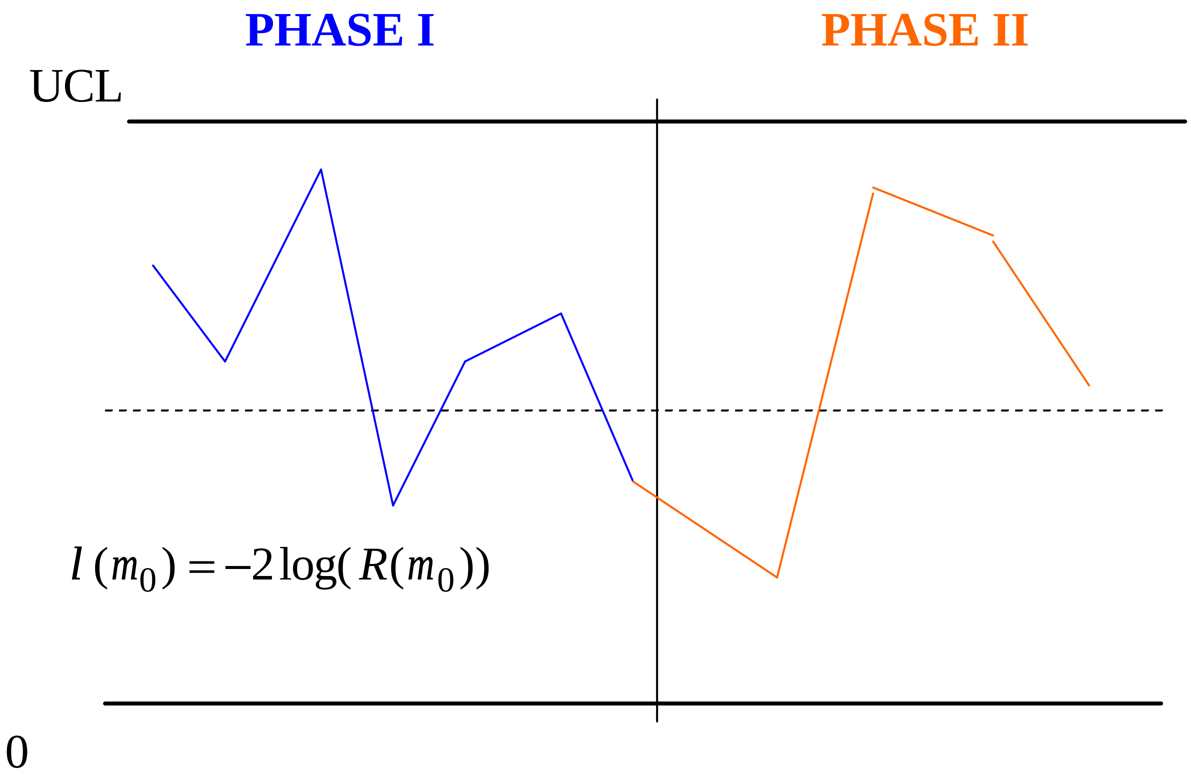
$$l_{(1)}(m_0), \ldots, l_{(T)}(m_0)$$

and then the percentile of interest will be

$$l_{(T(1-a))}(m_0).$$

- An alternative method is based on the ***Kernel smoother***: we first make the empirical distribution of $l(m_0)$ continuous by the smoother and then we take the $100(1-a)$-th percentile of the resulting distribution.

- The ***EL control chart*** is obtaining by representing the values of $l(m_0)$ for any observed sample in phase II. This control chart should not depend on the assumption of normality.



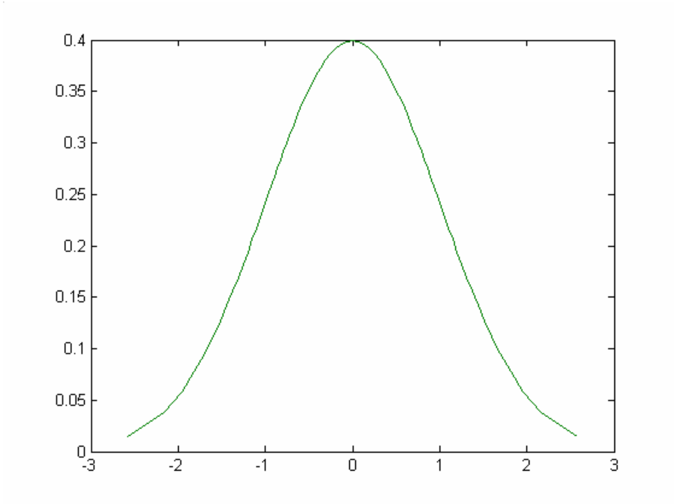
Performance comparison

- We carried on a ***simulation study*** to assess the performance of the proposed control chart with the X-bar chart and the bootstrap chart.

- The simulation is based on $C = 100$ charts ***generated*** under several distributions and with:
  - sample size: $n = 5$
  - number of observations in phase I: $mn = 20, 50, 100$
  - number of samples in phase II: $l = 10{,}000$

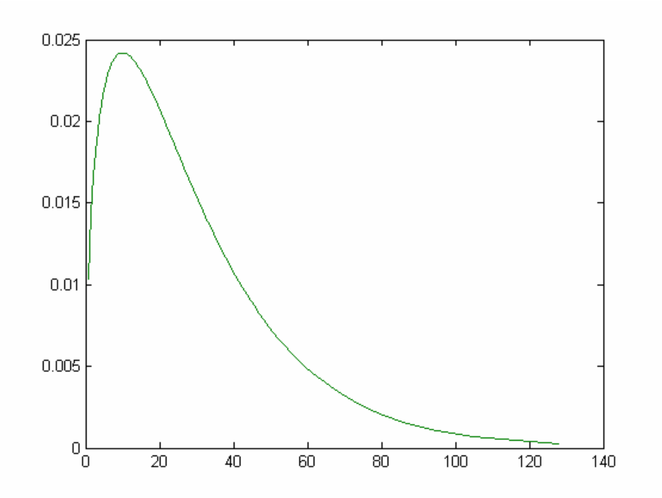
- The value of the ***type I error*** we considered is $a = 0.005$.

Standard Normal distribution $N(0,1)$



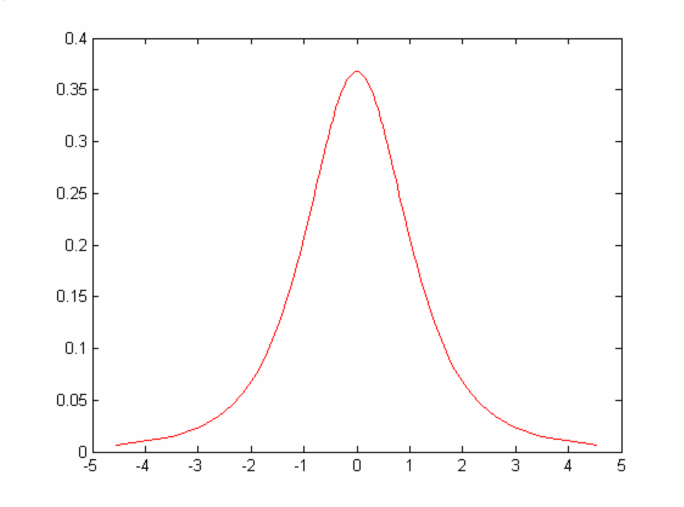
	Type of chart			
mn	X-bar	Bootstrap	EL	EL-Kernel
20	0.0235	0.0319	0.0172	0.0085
50	0.0101	0.0137	0.0083	0.0043
100	0.0075	0.0094	0.0068	0.0068

Gamma distribution $G(1.5, 20)$



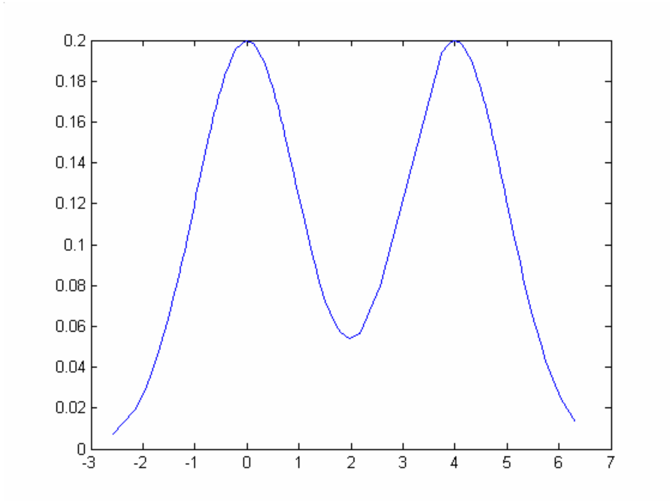
	Type of chart			
mn	X-bar	Bootstrap	EL	EL-Kernel
20	0.0508	0.0520	0.0301	0.0298
50	0.0354	0.0293	0.0206	0.0205
100	0.0239	0.0203	0.0171	0.0170

t-Student distribution $t(3)$



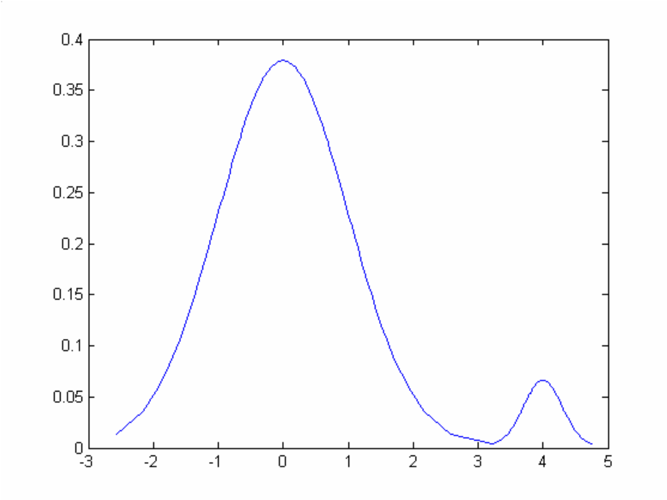
	Type of chart			
mn	X-bar	Bootstrap	EL	EL-Kernel
20	0.0447	0.0658	0.0169	0.0073
50	0.0238	0.0290	0.0086	0.0037
100	0.0212	0.0212	0.0082	0.0081

Symmetric bimodal distribution 1/2 N(0,1) +1/2 N(4,1)



	Type of chart			
mn	X-bar	Bootstrap	EL	EL-Kernel
20	0.0163	0.0195	0.0114	0.0018
50	0.0070	0.0100	0.0083	0.0030
100	0.0041	0.0079	0.0065	0.0064

Skewed bimodal distribution $0.95\,N(0,1) + 0.05\,N(4, 1/3))$



	Type of chart			
mn	X-bar	Bootstrap	EL	EL-Kernel
20	0.0388	0.0151	0.0153	0.0073
50	0.0136	0.0132	0.0076	0.0023
100	0.0112	0.0102	0.0074	0.0074

Conclusions

- Bootstrap chart is usually the worst option.

- EL and EL-Kernel charts usually behave better than the other two charts and so we suggest the use of the proposed approach especially when we are aware of the distribution of the variable of interest.

- The improvement of Kernel for EL is worthwhile, especially when the number of observations in Phase I (mn) is small, 20 say.

- The most difficult situation to deal with seems to be the Gamma one. In this case no chart works properly; this is probably due to the high skewness and kurtosis of the distribution. However, even in this case, EL based charts behave better than X-bar and Bootstrap charts.

Further developments

- Our aim for the future is that extending the proposed approach to
    - control population variability (univariate case)
    - deal with mean vectors and covariance matrices