Analysis of binary panel data by static and dynamic logit models

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Preliminaries

- Longitudinal (or panel) data consist of *repeated observations on the some subjects at different occasions*
- Data of this type are commonly used in many fields, especially in *economics* (e.g. analysis of labor market, analysis of the customer behavior) and in *medicine* (e.g. study of aging, efficacy of a drug)
- Many longitudinal datasets are now available:
  - National Longitudinal Surveys of Labor Market Experience (NLS)
  - Panel Study of Income Dynamics (PSID)
  - European Community Household Panel (ECHP)
  - The Netherlands Socio-Economic Panel (SEP)
  - German Social Economic Panel (GSOEP)
  - British Household Panel Survey (BHPS)

- With respect to cross-sectional data, longitudinal data have the advantage of allowing one to study (or to take into account in a natural way):
  - *unobserved heterogeneity*
  - *dynamic relationships*
  - *causal effects*
- Longitudinal studies suffer from *attrition*
- We will study, in particular, models for the analysis of *binary* response variables

Basic notation

- There are n *subjects* (or individuals) in the sample, with:
  - T_i: number of occasions at which subject i is observed
  - y_{it}: response variable (binary or categorical) for subject i at occasion t
  - $\mathbf{x}_{it}$: vector of covariates for subject i at occasion t
- The dataset is said *balanced* if all subjects are observed at the same occasions ($T_1 = \cdots = T_n$); otherwise, it is said *unbalanced*
- Usually, the dataset is unbalanced because of *attrition*; particular care is needed in this case, especially when the non-responses are not ignorable
- For simplicity, we will usually refer to the *balanced case* and we will denote by T the number of occasions (common to all subjects)

Example (similar to Hyslop, 1999)

- We consider a sample of $n = 1908$ women, aged 19 to 59 in 1980, who were followed from 1979 to 1985 (source *PSID*)
- *Response variable*: y_{it} equal to 1 if woman i has a job position during year t and to 0 otherwise
- *Covariates:*
  - age in 1980 (time-constant)
  - race (dummy equal to 1 for a black; time-constant)
  - educational level (number of year of schooling; time-constant)
  - number of children aged 0 to 2 (time-varying), aged 3 to 5 (time-varying) and aged 6 to 17 (time-varying)
  - permanent income (average income of the husband from 1980 to 1985; time-constant)
  - temporary income (difference between income of the husband in a year and permanent income; time-varying)

Homogeneous static logit and probit models

- These are *simple models* for the probabilities

$$\pi(\mathbf{x}_{it}) = p(y_{it} = 1 \mid \mathbf{x}_{it})$$

- These probabilities are modeled so that they always belong to [0,1]; this is obtained by a *link function* of type logit or probit:

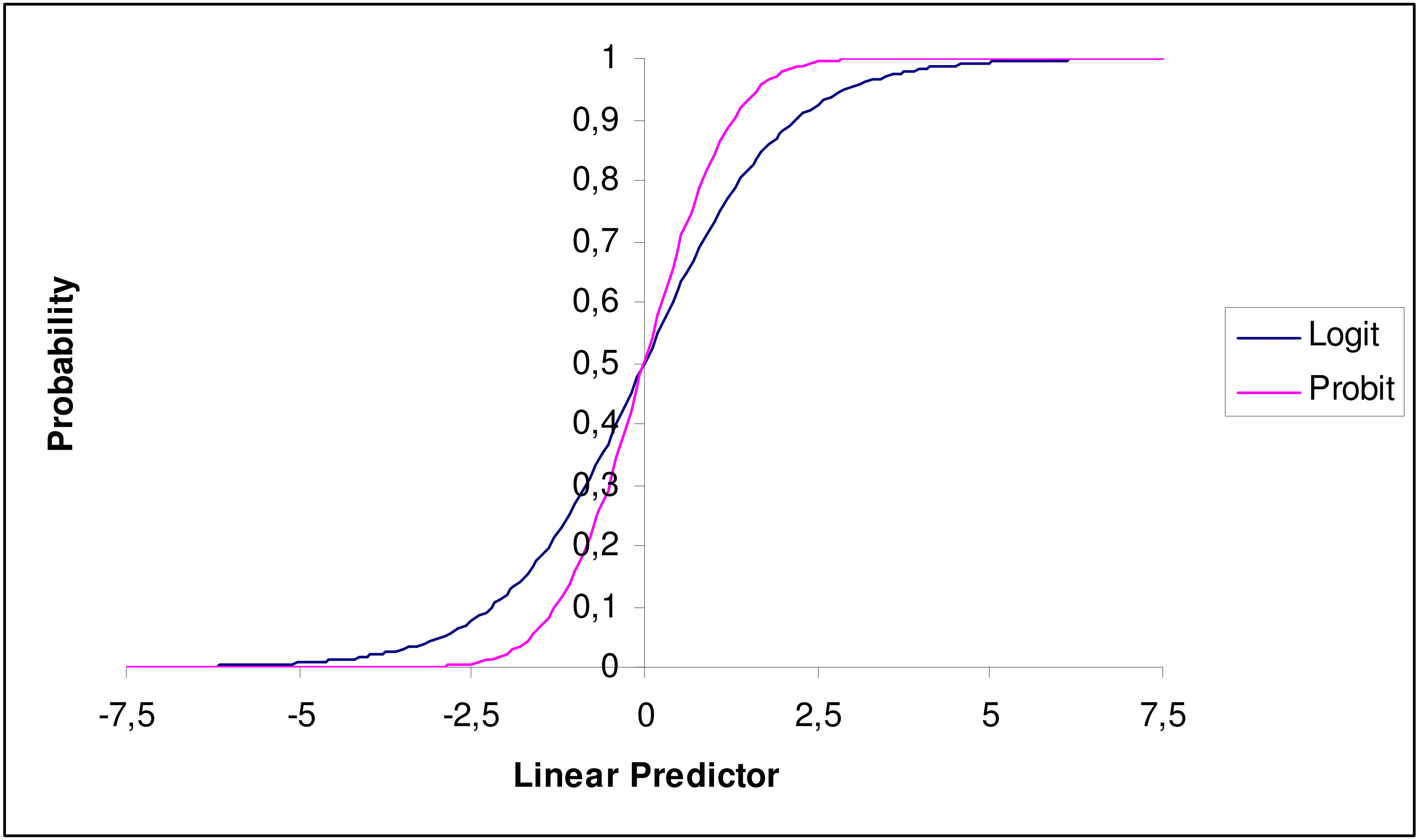
    - logit: $$\log \frac{\pi(\mathbf{x}_{it})}{1-\pi(\mathbf{x}_{it})} = \mathbf{x}_{it}'\boldsymbol{\beta}$$

    - probit: $$\Phi^{-1}[\pi(\mathbf{x}_{it})] = \mathbf{x}_{it}'\boldsymbol{\beta}$$

    - $\Phi^{-1}(\cdot)$: inverse of the distribution function of the standard normal distribution

- The *inverse link function* is:
  - logit: $$\pi(\mathbf{x}_{it}) = \frac{\exp(\mathbf{x}_{it}'\boldsymbol{\beta})}{1+\exp(\mathbf{x}_{it}'\boldsymbol{\beta})}$$
  - probit: $$\pi(\mathbf{x}_{it}) = \Phi(\mathbf{x}_{it}'\boldsymbol{\beta})$$
  - $\Phi(\cdot)$: distribution function of the standard normal distribution

- Other *basic assumptions* of the models:
  - Independence between the response variables given the covariates (*static models*)
  - The heterogeneity between subjects is only explained on the basis of observable covariates and then unobserved heterogeneity is ruled out (*homogeneous models*)


1
0,9
0,8
0,7
0,6
0,5
0,4
0,3
0,2
0,1
0
Probability
-7,5
-5
-2,5
0
2,5
5
7,5
Linear Predictor
Logit
Probit

Threshold model

- Logit and probit models may be interpreted on the basis of an *underlying linear model* for the propensity to experience a certain situation:

$$y_{it}^{*} = \mathbf{x}_{it}'\boldsymbol{\beta} + \varepsilon_{it}$$

  - ε_{it}: error term with standard normal or logistic distribution

- The situation is experienced ($y_{it} = 1$) only if $y_{it}^{*} \geq 0$ (*threshold*), i.e.

$$y_{it} = 1(y_{it}^{*} \geq 0) = \begin{cases} 1 & \text{if } y_{it}^{*} \geq 0 \\ 0 & \text{if } y_{it}^{*} < 0 \end{cases}$$

- Since the distribution of ε_{it} is symmetric, we have that

$$p(y_{it} = 1 \mid \mathbf{x}_{it}) = p(y_{it}^{*} \geq 0 \mid \mathbf{x}_{it}) = p(\mathbf{x}_{it}'\boldsymbol{\beta} \geq -\varepsilon_{it} \mid \mathbf{x}_{it}) = p(\varepsilon_{it} \leq \mathbf{x}_{it}'\boldsymbol{\beta} \mid \mathbf{x}_{it})$$

corresponding to the *logistic or standard normal distr. function*

Model estimation

- The most used method to fit logit and probit models is the *maximum likelihood method*, which is based on the maximization of the log-likelihood:

$$L(\boldsymbol{\beta}) = \sum_i \sum_t y_{it} \log[\pi(\mathbf{x}_{it})] + (1 - y_{it}) \log[1 - \pi(\mathbf{x}_{it})]$$

- Maximization of $L(\boldsymbol{\beta})$ can be performed by the *Newton-Raphson algorithm*. Starting from an initial estimate $\boldsymbol{\beta}^{(0)}$, the algorithm consists of updating the estimate at step h as

$$\boldsymbol{\beta}^{(h)} = \boldsymbol{\beta}^{(h-1)} + \mathbf{J}\left(\boldsymbol{\beta}^{(h-1)}\right)^{-1} \mathbf{s}\left(\boldsymbol{\beta}^{(h-1)}\right)$$

  - $\mathbf{s}(\boldsymbol{\beta}) = \dfrac{\partial L(\boldsymbol{\beta})}{\partial \boldsymbol{\beta}}$: *score vector*
  - $\mathbf{J}(\boldsymbol{\beta}) = -\dfrac{\partial^2 L(\boldsymbol{\beta})}{\partial \boldsymbol{\beta} \partial \boldsymbol{\beta}'}$: *observed information matrix*

- An alternative algorithm is the *Fisher-scoring* which uses the *expected information matrix*

$$\mathbf{I}(\boldsymbol{\beta}) = -E\left(\frac{\partial^2 L(\boldsymbol{\beta})}{\partial\boldsymbol{\beta}\partial\boldsymbol{\beta}'}\right)$$

  instead of the observed information matrix

- *Standard errors* for each element of $\hat{\boldsymbol{\beta}}$ is computed as the square root of the corresponding diagonal element of $\mathbf{I}(\boldsymbol{\beta})^{-1}$

- For the *logit model* we have

$$L(\boldsymbol{\beta}) = \sum_i \sum_t y_{it}\mathbf{x}_{it}'\boldsymbol{\beta} - \log[1 + \exp(\mathbf{x}_{it}'\boldsymbol{\beta})]$$

$$\mathbf{s}(\boldsymbol{\beta}) = \sum_i \sum_t [y_{it} - \pi(\mathbf{x}_{it})]\mathbf{x}_{it} ,$$

$$\mathbf{J}(\boldsymbol{\beta}) = \mathbf{I}(\boldsymbol{\beta}) = \sum_i \sum_t \pi(\mathbf{x}_{it})[1 - \pi(\mathbf{x}_{it})]\mathbf{x}_{it}\mathbf{x}_{it}'$$

Example

- Maximum likelihood estimates for the PSID dataset (*logit model*)

Parameter	Estimate	*s.e.*	*t*-statistic	*p*-value
Intercept	-0.6329	0.3093	-2.0464	0.0407
Age	0.0923	0.0172	5.3750	0.0000
Age^2/100	-0.1694	0.0221	-7.6496	0.0000
Race	0.3161	0.0517	6.1188	0.0000
Education	0.3278	0.0152	21.5510	0.0000
Kids 0-2	-0.7810	0.0447	-17.4890	0.0000
Kids 3-5	-0.6450	0.0406	-15.8920	0.0000
Kids 6-17	-0.1400	0.0201	-6.9497	0.0000
Perm. inc.	-0.0215	0.0014	-15.5820	0.0000
Temp. inc.	-0.0070	0.0023	-2.9860	0.0028

- Maximum likelihood estimates for the PSID dataset (*probit model*)

Parameter	**Estimate**	***s.e.***	***t*-statistic**	***p*-value**
Intercept	-0.3770	0.1843	-2.0451	0.0408
Age	0.0548	0.0103	5.3308	0.0000
Age^2/100	-0.1009	0.0133	-7.5893	0.0000
Race	0.1990	0.0304	6.5362	0.0000
Education	0.1921	0.0089	21.6380	0.0000
Kids 0-2	-0.4666	0.0266	-17.5360	0.0000
Kids 3-5	-0.3871	0.0242	-15.9790	0.0000
Kids 6-17	-0.0846	0.0120	-7.0353	0.0000
Perm. inc.	-0.0115	0.0008	-14.3010	0.0000
Temp. inc.	-0.0027	0.0013	-2.0524	0.0401

- By a general rule the estimate of $\boldsymbol{\beta}$ under the logit model is approx. equal to 1.6 times the estimate of $\boldsymbol{\beta}$ under the probit model

Heterogeneous static logit and probit models

- A method to incorporate unobserved heterogeneity in a logit or probit model is to include a set of *subject-specific parameters* α_i and then assuming that

$$\pi(\alpha_i, \mathbf{x}_{it}) = \frac{\exp(\alpha_i + \mathbf{x}_{it}'\boldsymbol{\beta})}{1+\exp(\alpha_i + \mathbf{x}_{it}'\boldsymbol{\beta})} \quad \text{or} \quad \pi(\alpha_i, \mathbf{x}_{it}) = \Phi(\alpha_i + \mathbf{x}_{it}'\boldsymbol{\beta})$$

  - $\pi(\alpha_i, \mathbf{x}_{it}) = p(y_{it} = 1 | \alpha_i, \mathbf{x}_{it})$: conditional probability of success given α_i and $\mathbf{x}_{it}$

- The parameters α_i may be treated as fixed or random:
  - *fixed*: the response variables y_{it} are still assumed independent
  - *random*: the response variables y_{it} are assumed conditionally independent given α_i

- The most used *estimation methods* of the model are:
  - joint maximum likelihood (fixed-parameters)
  - conditional maximum likelihood (only for the logit model)
  - marginal maximum likelihood (random-parameters)

Joint maximum likelihood (JML) method

- It consists of maximizing the log-likelihood

$$L(\boldsymbol{\alpha},\boldsymbol{\beta})=\sum_i\sum_t y_{it}\log[\pi(\alpha_i,\mathbf{x}_{it})]+(1-y_{it})\log[1-\pi(\alpha_i,\mathbf{x}_{it})]$$

  with respect to (*jointly*) $\boldsymbol{\alpha}=(\alpha_1,\ldots,\alpha_n)'$ and $\boldsymbol{\beta}$

- The method is simple to implement for both logit and probit models

- It is usually based on an iterative algorithm which alternates Newton-Raphson (or Fisher scoring) steps for updating the estimate of each α_i with Newton-Raphson (or Fisher scoring) steps for updating the estimate of $\boldsymbol{\beta}$

- The JML estimator:
  - *does not exist* (for α_i) when $y_{i+} = 0$ or $y_{i+} = T$, with $y_{i+} = \sum_t y_{it}$
  - is not consistent with T fixed as n grows to infinity and so a JML estimate is not reliable for small T even if n is very large; this is because the number of parameters increases with n (*incidental parameters problem*; Neyman and Scott, 1948)
- For the heterogeneous logit model we must *solve the equations*:

$$\frac{\partial L(\boldsymbol{\beta})}{\partial \alpha_i} = \sum_t [y_{it} - \pi(\alpha_i, \mathbf{x}_{it})] = 0, \qquad i = 1, \ldots, n$$

$$\frac{\partial L(\boldsymbol{\beta})}{\partial \boldsymbol{\beta}} = \sum_i \sum_t [y_{it} - \pi(\alpha_i, \mathbf{x}_{it})]\mathbf{x}_{it} = \mathbf{0}$$

Conditional maximum likelihood (CML) method

- This estimation method may be used only for the *logit model*

- For the logit model we have that, for $i=1,\dots,n$, y_{i+} is a *sufficient statistic* for the subject specific-parameter α_i and, consequently, we can construct a conditional likelihood which does not depend on these parameters but only on $\boldsymbol{\beta}$

- The *conditional log-likelihood* may be expressed as

$$L_c(\boldsymbol{\beta}) = \sum_i \log[p(\mathbf{y}_i \mid \mathbf{X}_i, y_{i+})], \qquad \mathbf{y}_i = (y_{i1},\dots,y_{iT})'$$

- From the maximization of $L_c(\boldsymbol{\beta})$ we obtain the *CML estimator* of $\boldsymbol{\beta}$, $\hat{\boldsymbol{\beta}}_c$, which is consistent for fixed T as n grows to infinity; this maximization may be performed on the basis of a *Newton-Raphson* algorithm which also produces standard errors for $\hat{\boldsymbol{\beta}}_c$

- An important *drawback*, common to all fixed-parameters approaches, is that the regression parameters for the time-constant covariates are not estimable

- The *probability of the response configuration* $\mathbf{y}_i$ may be expressed as

$$p(\mathbf{y}_i \mid \mathbf{X}_i) = \prod_t \frac{\exp[y_{it}(\alpha_i + \mathbf{x}_{it}'\boldsymbol{\beta})]}{1+\exp(\alpha_i + \mathbf{x}_{it}'\boldsymbol{\beta})} = \frac{\exp(y_{i+}\alpha_i + \sum_t y_{it}\mathbf{x}_{it}'\boldsymbol{\beta})}{\prod_t [1+\exp(\alpha_i + \mathbf{x}_{it}'\boldsymbol{\beta})]}$$

- The *probability of the sum of the responses* y_{i+} is then equal to

$$p(y_{i+} \mid \mathbf{X}_i) = \frac{\exp(y_{i+}\alpha_i)}{\prod_t [1+\exp(\alpha_i + \mathbf{x}_{it}'\boldsymbol{\beta})]} \sum_{\mathbf{z}(y_{i+})} \exp(\textstyle\sum_t z_t \mathbf{x}_{it}'\boldsymbol{\beta})$$

  - $\sum_{\mathbf{z}(y_{i+})}$ is extended to all the response configurations $\mathbf{z} = (z_1, \ldots, z_T)'$ with sum $z_+ = y_{i+}$

- The *conditional probability of the response configuration* $\mathbf{y}_i$ given y_{i+} is then

$$p(\mathbf{y}_i \mid \mathbf{X}_i, y_{i+}) = \frac{\exp(\sum_t y_{it}\mathbf{x}_{it}'\boldsymbol{\beta})}{\sum_{\mathbf{z}(y_{i+})}\exp(\sum_t z_t \mathbf{x}_{it}'\boldsymbol{\beta})}$$

which is equal to 1 for $y_{i+} = 0$ or $y_{i+} = T$ regardless of the value of $\boldsymbol{\beta}$

- The *conditional log-likelihood* is equal to

$$L_c(\boldsymbol{\beta}) = \sum_i 1(0 < y_{i+} < T)\left\{\sum_t y_{it}\mathbf{x}_{it}'\boldsymbol{\beta} - \log\left[\sum_{\mathbf{z}(y_{i+})}\exp(\sum_t z_t \mathbf{x}_{it}'\boldsymbol{\beta})\right]\right\}$$

- *Score* and observed *information matrix*, to be used within the Newton-Raphson algorithm and to compute the standard errors for $\hat{\boldsymbol{\beta}}_c$:

$$\mathbf{s}_c(\boldsymbol{\beta}) = \frac{\partial L_c(\boldsymbol{\beta})}{\partial \boldsymbol{\beta}} = \sum_i 1(0 < y_{i+} < T)\mathbf{X}_i'\left[\mathbf{y}_i - \mathrm{E}_{\boldsymbol{\beta}}(\mathbf{y}_i \mid y_{i+})\right]$$

$$\mathbf{J}_c(\boldsymbol{\beta}) = -\frac{\partial L_c^2(\boldsymbol{\beta})}{\partial \boldsymbol{\beta}\partial \boldsymbol{\beta}'} = \sum_i 1(0 < y_{i+} < T)\mathbf{X}_i'\mathrm{V}_{\boldsymbol{\beta}}(\mathbf{y}_i \mid y_{i+})\mathbf{X}_i$$

- JML and CML estimates for the PSID dataset (*logit model*)

Parameter	**JML estimate**	**CML estimate**	***s.e.***	***t-statistic***	***p*-value**
Intercept	-	-	-	-	-
Age	-	-	-	-	-
Age^2/100	-	-	-	-	-
Race	-	-	-	-	-
Education	-	-	-	-	-
Kids 0-2	-1.3660	-1.1537	0.0899	-12.8290	0.0000
Kids 3-5	-0.9912	-0.8373	0.0840	-9.9638	0.0000
Kids 6-17	-0.2096	-0.1764	0.0637	-2.7691	0.0056
Perm. inc.	-	-	-	-	-
Temp. inc.	-0.0162	-0.0136	0.0033	-4.1186	0.0000

(-) not estimable

Marginal maximum likelihood (MML)

- This estimation method may be used for both *logit and probit models*

- It is based on the assumption that the subject-specific parameters α_i are *random parameters* with the same distribution $f(\alpha_i)$ which is independent of $\mathbf{X}_i$

- It is also assumed that the response variables $y_{i1},\ldots,y_{iT}$ are *conditionally independent* given α_i, so that

$$p(\mathbf{y}_i \mid \mathbf{X}_i) = \int p(\mathbf{y}_i \mid \alpha_i, \mathbf{X}_i) f(\alpha_i) d\alpha_i, \quad p(\mathbf{y}_i \mid \alpha_i, \mathbf{X}_i) = \prod_t p(y_{it} \mid \alpha_i, \mathbf{x}_{it}),$$

  where the integral must usually be computed by a numerical method (e.g. quadrature)

- The *marginal log-likelihood* is then

$$L_m(\boldsymbol{\beta}) = \sum_i \log[p(\mathbf{y}_i \mid \mathbf{X}_i)],$$

which can be maximized, with respect to $\boldsymbol{\beta}$ and (possibly) the parameters of the distribution of the random effects, by a Newton-Raphson algorithm

Logit model with normal random effects

- Under the assumption $\alpha_i \sim N(\mu, \sigma^2)$, for the *logit model* we have

$$p(\mathbf{y}_i \mid \mathbf{X}_i) = \int p(\mathbf{y}_i \mid w, \mathbf{X}_i)\phi(w)dw$$

  - $p(\mathbf{y}_i \mid w, \mathbf{X}_i) = \prod_t \frac{\exp[y_{it}(\mu + w\sigma + \mathbf{x}_{it}'\boldsymbol{\beta})]}{1+\exp(\mu + w\sigma + \mathbf{x}_{it}'\boldsymbol{\beta})} = \prod_t \frac{\exp\{y_{it}[\mathbf{z}_{it}(w)'\boldsymbol{\gamma}]\}}{1+\exp[\mathbf{z}_{it}(w)'\boldsymbol{\gamma}]}$
  - $\phi(w)$: density function of the standard normal distribution
  - $\mathbf{z}_{it}(w) = (1 \quad w \quad \mathbf{x}_{it}')', \qquad \boldsymbol{\gamma} = (\mu \quad \sigma \quad \boldsymbol{\beta}')'$

- The *score vector* and the (empirical) *information matrix* are given by

$$\mathbf{s}_m(\boldsymbol{\gamma}) = \frac{\partial L_m(\boldsymbol{\gamma})}{\partial \boldsymbol{\gamma}} = \sum_i \mathbf{s}_{m,i}(\boldsymbol{\gamma}), \qquad \mathbf{s}_{m,i}(\boldsymbol{\gamma}) = \frac{1}{p(\mathbf{y}_i \mid \mathbf{X}_i)} \int \frac{\partial p(\mathbf{y}_i \mid w, \mathbf{X}_i)}{\partial \boldsymbol{\gamma}} \phi(w)dw$$

$$\tilde{\mathbf{J}}_m(\boldsymbol{\gamma}) = \sum_i \mathbf{s}_{m,i}(\boldsymbol{\gamma})\mathbf{s}_{m,i}(\boldsymbol{\gamma})' - \frac{1}{n}\mathbf{s}_m(\boldsymbol{\gamma})\mathbf{s}_m(\boldsymbol{\gamma})'$$

Pros and cons of MML

- The MML method is more complicate to implement than fixed-effects methods (JML, CML), but it allows us to estimate the regression parameters for *both time-fixed and time-varying covariates*

- The MML also allows us to *predict future outcomes*

- Special care has to be used for the *specification of the distribution of the random effects*. It may be restrictive to assume:
  - a specific parametric function for these effects, such as the normal distribution
  - that the distribution does not depend on the covariates

- The approach may be extended to overcome these drawbacks:
  - a discrete distribution with free support points and mass probabilities may be used for the random effects; the approach is in this case of *latent class* type and requires the implementation of an EM algorithm (Dempster *et al.*, 1977) and the choice of the number of support points
  - the parameters of the distribution of the random effects are allowed to depend on the covariates; one possibility is the *correlated effect model* of Chamberlain (1984)

- JML, CML and MML-normal estimates for the PSID dataset (*logit model*); MML algorithm uses 51 quadrature points from –5 to 5

Parameter	**JML estimate**	**CML estimate**	**MML estimate**	***s.e.***	***t*-*statistic***	***p*-value**
Intercept	-	-	-2.9448	1.3461	-2.1876	0.0287
Std.dev (σ)			3.2196	0.1066	30.2090	0.0000
Age	-	-	0.2652	0.0712	3.7243	0.0002
Age^2/100	-	-	-0.4285	0.0906	-4.7271	0.0000
Race	-	-	0.6800	0.2162	3.1449	0.0017
Education	-	-	0.6737	0.0643	10.4810	0.0000
Kids 0-2	-1.3660	-1.1537	-1.3418	0.0773	-17.3490	0.0000
Kids 3-5	-0.9912	-0.8373	-1.0260	0.0635	-16.1680	0.0000
Kids 6-17	-0.2096	-0.1764	-0.2533	0.0438	-5.7775	0.0000
Perm. inc.	-	-	-0.0427	0.0036	-11.9610	0.0000
Temp. inc.	-0.0162	-0.0136	-0.0110	0.0023	-4.7554	0.0000

Summary of the models fit

- Estimates for the PSID dataset (*logit model*):

Method	Log-likelihood	n. parameters	AIC	BIC
Homogenous	-7507.3	10	15034.6	15090.1
Heterogeneous-JML	-2986.3	1912	9796.6	20415.5
Heterogeneous-CML*	-2128.5	4	4265.0	4287.2
Heterogeneous-MML-normal	-5264.4	11	10550.8	10611.9

(*) not directly comparable with the others

- AIC: *Akaike Information Criterion* (Akaike, 1973)

$$\mathrm{AIC} = -2(\mathrm{max.\,log-likelihood}) + 2(\mathrm{n.\,parameters})$$

- BIC: *Bayesian Information Criterion* (Schwarz, 1978)

$$\mathrm{BIC} = -2(\mathrm{max.\,log-likelihood}) + \log(n)(\mathrm{n.\,parameters})$$

Dynamic models

- Previous models are *static*: they do not include the lagged response variable among the regressors

- The *dynamic* version of these models is based on the assumption that, given $y_{i,t-1}$ and α_i, every y_{it} is conditionally independent of $y_{i1},\dots,y_{i,t-2}$ and that

$$\pi(\alpha_i, \mathbf{x}_{it}, y_{i,t-1}) = \frac{\exp(\alpha_i + \mathbf{x}_{it}'\boldsymbol{\beta} + y_{i,t-1}\gamma)}{1+\exp(\alpha_i + \mathbf{x}_{it}'\boldsymbol{\beta} + y_{i,t-1}\gamma)}$$

or

$$\pi(\alpha_i, \mathbf{x}_{it}, y_{i,t-1}) = \Phi(\alpha_i + \mathbf{x}_{it}'\boldsymbol{\beta} + y_{i,t-1}\gamma)$$

➢ $\pi(\alpha_i, \mathbf{x}_{it}, y_{i,t-1}) = p(y_{it} = 1 \mid \alpha_i, \mathbf{x}_{it}, y_{i,t-1})$: conditional probability of success

- The initial observation y_{i0} must be known. When the parameters α_i are random, the *initial condition problem* arises. The simplest approach, which however can lead to an biased estimator of $\boldsymbol{\beta}$ and γ, is to treat y_{i0} as an exogenous covariate

- Dynamic models have the great advantage of allowing us to distinguish between:
  - *true state dependence* (Heckman, 1981): effect that experimenting a certain situation in the present has on the propensity of experimenting the same situation in the future
  - *spurious state dependence*: propensity common to all occasions which is measured by α_i and the time-constant covariates

Estimation of dynamic models

- The subject-specific parameters α_i may be considered as *fixed* or *random*

- With *fixed parameters* α_i, the conditional probability of a response configuration $\mathbf{y}_i$ given y_{i0} is:

$$p(\mathbf{y}_i \mid \alpha_i, \mathbf{X}_i, y_{i0}) = \prod_t p(y_{it} \mid \alpha_i, \mathbf{x}_{it}, y_{i,t-1})$$

- The *random-parameters* approach requires to formulate a distribution for the parameters α_i, so that

$$p(\mathbf{y}_i \mid \mathbf{X}_i, y_{i0}) = \int p(\mathbf{y}_i \mid \alpha_i, \mathbf{X}_i, y_{i0}) f(\alpha_i) d\alpha_i,$$

$$p(\mathbf{y}_i \mid \alpha_i, \mathbf{X}_i, y_{i0}) = \prod_t p(y_{it} \mid \alpha_i, \mathbf{x}_{it}, y_{i,t-1})$$

- The most used *estimation methods* for dynamic models are the same as for static models:
  - joint maximum likelihood (fixed-parameters)
  - conditional maximum likelihood (only for the logit model)
  - marginal maximum likelihood (random-parameters)

Joint maximum likelihood (JML) method

- The *log-likelihood* has again a simple form:

$$L(\boldsymbol{\alpha},\boldsymbol{\beta},\gamma)=\sum_i\sum_t y_{it}\log[\pi(\alpha_i,\mathbf{x}_{it},y_{i,t-1})]+(1-y_{it})\log[1-\pi(\alpha_i,\mathbf{x}_{it},y_{i,t-1})]$$

  and must be jointly maximized with respect to $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ and γ

- Maximizing the *log-likelihood* may be performed by using a Newton-Raphson (or Fisher scoring) algorithm which alternates a step in which the estimate of each parameter α_i is updated with a step in which the estimates of $\boldsymbol{\beta}$ and γ are updated

- The algorithm is essentially the same as that used for static models, but with $y_{i,t-1}$ included among the covariates $\mathbf{x}_{it}$

- The JML estimator has the same drawbacks it has for static models:
  - it *does not exist* (for α_i) when $y_{i+}=0$ or $y_{i+}=T$, with $y_{i+}=\sum_t y_{it}$
  - it is *not consistent* with T fixed as n grows to infinity

- JML estimates for the PSID dataset (*static and dynamic logit models*)

Parameter	Static logit	Dynamic logit
Kids 0-2	-1.3660	-1.2688
Kids 3-5	-0.9912	-0.8227
Kids 6-17	-0.2096	-0.1730
Temp. inc.	-0.0162	-0.0112
Lagged response	-	0.5696

- A positive state dependence is observed and the *fit of the logit model* improves considerably by including the lagged response variable

Model	Log-likelihood	n. parameters	AIC	BIC
Static logit	-2986.3	1912	9796.6	20415.5
Dynamic logit	-2317.9	1913	8461.8	19086.2

Conditional maximum likelihood (CML) method

- The CML method may be used to estimate the dynamic logit model *only in particular circumstances*

- Under these circumstances, the method is difficult to implement since the sum of the response variables y_{i+} is not a *sufficient statistic* for the subject specific-parameter α_i

- The CML approach may be used when $T = 3$ and there are *no covariates*, so that

$$p(\mathbf{y}_i \mid \alpha_i, y_{i0}) = \frac{\exp(y_{i+}\alpha_i + y_{i*}\gamma)}{\prod_t [1 + \exp(y_{it}\alpha_i + y_{i,t-1}\gamma)]}, \qquad y_{i*} = \sum_t y_{i,t-1} y_{it}$$

- The response configurations $\mathbf{y}_i = (0 \quad 1 \quad y_{i3})'$ and $\mathbf{y}_i = (1 \quad 0 \quad y_{i3})'$ have *conditional probability*

$$p[(0 \quad 1 \quad y_{i3})' | \alpha_i, y_{i0}] = \frac{\exp[(1+y_{i3})\alpha_i + y_{i3}\gamma]}{[1+\exp(\alpha_i + y_{i0}\gamma)][1+\exp(\alpha_i)][1+\exp(\alpha_i+\gamma)]}$$

$$p[(1 \quad 0 \quad y_{i3})' | \alpha_i, y_{i0}] = \frac{\exp[(1+y_{i3})\alpha_i + y_{i0}\gamma]}{[1+\exp(\alpha_i + y_{i0}\gamma)][1+\exp(\alpha_i+\gamma)][1+\exp(\gamma)]}$$

- We can then condition on $y_{i0}, y_{i1} + y_{i2} = 1, y_{i3}$ obtaining the *conditional probabilities*

$$p[(0 \quad 1 \quad y_{i3}) | \alpha_i, y_{i0}, y_{i1} + y_{i2} = 1, y_{i3}] = \frac{\exp(y_{i3}\gamma)}{\exp(y_{i3}\gamma) + \exp(y_{i0}\gamma)} = \frac{1}{1+\exp[(y_{i0} - y_{i3})\gamma]}$$

$$p[(1 \quad 0 \quad y_{i3}) | \alpha_i, y_{i0}, y_{i1} + y_{i2} = 1, y_{i3}] = \frac{\exp(y_{i0}\gamma)}{\exp(y_{i3}\gamma) + \exp(y_{i0}\gamma)} = \frac{\exp[(y_{i0} - y_{i3})\gamma]}{1+\exp[(y_{i0} - y_{i3})\gamma]}$$

- The corresponding *conditional log-likelihood* is

$$L_c(\gamma) = \sum_i d_i (y_{i1}(y_{i0} - y_{i3})\gamma - \log\{1 + \exp[(y_{i0} - y_{i3})\gamma]\})$$

$$d_i = 1(y_{i1} + y_{i2} = 1),$$

  which may be maximized by a simple Newton-Raphson algorithm; it results a consistent estimator of γ (Chamberlain, 1993)

- The conditional approach may also be implemented for $T > 3$ on the basis of the *pairwise conditional log-likelihood*

$$L_{pc}(\gamma) = \sum_i \sum_{s<t<T} 1(y_{is} + y_{it} = 1)(y_{is}(y_{i,s-1} - y_{i,t+1})\gamma - \log\{1 + \exp[(y_{i,s-1} - y_{i,t+1})\gamma]\})$$

  the resulting estimator has the same properties it has for $T = 3$ and, in particular, it is consistent for T fixed as n grows to infinity

- The conditional approach may also be used in the presence of covariates, provided that:
    - the probability that each *discrete covariate* is time-constant is positive (this rules out the possibility of time dummies)
    - the support of the distribution of the *continuous* covariates satisfies suitable conditions
- The *algorithm* to be implemented in this case is rather complicate and leads to a consistent estimator of $\boldsymbol{\beta}$ and γ which, however, is not $\sqrt{n}$-consistent (Honoré and Kyriazidou, 2000)
- The CML approach has the advantage, over the MML approach, of not requiring to formulate the *distribution of the subject-specific parameters*. It also does not suffer from the *initial condition problem* and y_{i0} may be treated as an exogenous covariate

Marginal maximum likelihood (MML) method

- This estimation method may be used for both *dynamic logit and probit models*

- The algorithm is essentially the same as that for static models, but we have to use an *extended vector of covariates* which includes the lagged response variable

- For the *dynamic logit model with normal random effects* we have to maximize

$$L_m(\tilde{\boldsymbol{\gamma}}) = \sum_i p(\mathbf{y}_i \mid \mathbf{X}_i, y_{i0}), \qquad p(\mathbf{y}_i \mid \mathbf{X}_i, y_{i0}) = \int p(\mathbf{y}_i \mid w, \mathbf{X}_i, y_{i0}) \phi(w) dw,$$

  - $p(\mathbf{y}_i \mid w, \mathbf{X}_i, y_{i0}) = \prod_t \dfrac{\exp[y_{it}(\mu + w\sigma + \mathbf{x}_{it}'\boldsymbol{\beta} + y_{i,t-1}\gamma)]}{1 + \exp(\mu + w\sigma + \mathbf{x}_{it}'\boldsymbol{\beta} + y_{i,t-1}\gamma)}$

- MML-normal estimates for the PSID dataset (*static and dynamic logit models*)

Parameter	**Static logit**	**Dynamic logit**	***s.e.***	***t-statistic***	***p*-value**
Intercept	-2.9448	-2.3313	0.6609	-3.5275	0.0004
Std.dev (σ)	3.2196	1.1352	0.0930	12.2060	0.0000
Age	0.2652	0.1037	0.0360	2.8820	0.0040
Age^2/100	-0.4285	-0.1813	0.0464	-3.9096	0.0001
Race	0.6800	0.3011	0.1054	2.8573	0.0043
Education	0.6737	0.3034	0.0332	9.1456	0.0000
Kids 0-2	-1.3418	-0.8832	0.0825	-10.7010	0.0000
Kids 3-5	-1.0260	-0.4390	0.0736	-5.9629	0.0000
Kids 6-17	-0.2533	-0.0819	0.0393	-2.0831	0.0372
Perm. inc.	-0.0427	-0.0189	0.0019	-10.1030	0.0000
Temp. inc.	-0.0110	-0.0036	0.0030	-1.1783	0.2387
Lagged response	-	2.7974	0.0653	42.8420	0.0000

- For the above example, a much *stronger state dependence effect* is observed with the MML method with respect to the JML method ($\hat{\gamma} = 2.7974$ vs. $\hat{\gamma} = 0.5696$)

- The suspect is that with the MML method the parameter γ is overestimated and this is because the assumptions on the distribution of the parameters α_i are *restrictive*

- A simple way to *give more flexibility* to the approach is to allow the mean of the normal distribution assumed on the parameters α_i to depend (through a linear regression model) on the initial observation y_{i0} and the corresponding time-varying covariates $\mathbf{x}_{i0}$

- MML-normal estimates for the PSID dataset (*dynamic and extended dynamic logit models*)

Parameter	**Dynamic logit**	**Exteded dynamic logit**	***s.e.***	***t*-statistic**	***p*-value**
Intercept	-2.3313	-3.4484	0.8942	-3.8566	0.0001
Std.dev (σ)	1.1352	1.6473	0.0900	18.2930	0.0000
Age	0.1037	0.1103	0.0502	2.1970	0.0280
Age^2/100	-0.1813	-0.1902	0.0647	-2.9410	0.0033
Race	0.3011	0.2744	0.1374	1.9971	0.0458
Education	0.3034	0.2864	0.0419	6.8412	0.0000
Kids 0-2	-0.8832	-1.0498	0.0917	-11.4470	0.0000
Kids 3-5	-0.4390	-0.5865	0.0871	-6.7369	0.0000
Kids 6-17	-0.0819	-0.1213	0.0624	-1.9426	0.0521
Perm. inc.	-0.0189	-0.0164	0.0031	-5.3094	0.0000
Temp. inc.	-0.0036	-0.0049	0.0032	-1.5133	0.1302
Lagged response	2.7974	1.8165	0.0824	22.0550	0.0000

- The estimate for the *state dependence effect seems now more reliable* ($\hat{\gamma} = 1.8164$ vs. $\hat{\gamma} = 2.7974$) even if it is strongly positive
- Estimates of the parameters for the mean of the distribution for α_i

Parameter	Estimate	*s.e.*	*t-statistic*	*p*-value
Kids 0-2	0.2669	0.1284	2.0787	0.0376
Kids 3-5	0.2424	0.1221	1.9864	0.0470
Kids 6-17	0.1299	0.0680	1.9102	0.0561
Temp. inc.	0.0116	0.0058	2.0180	0.0436
Initial observation (y_{i0})	2.5915	0.1586	16.3450	0.0000

Model	Log-likelihood	n. parameters	AIC	BIC
JML	-2317.9	1913	8461.8	19086.2
MML	-4188.1	12	8400.2	8466.8
MML extended	-3976.2	17	7986.4	8080.8

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