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Source: *The American Mathematical Monthly*, Jan., 1961, Vol. 68, No. 1 (Jan., 1961), pp. 1-17

Published by: Taylor & Francis, Ltd. on behalf of the Mathematical Association of America

Stable URL: https://www.jstor.org/stable/2311357

HOW TO CUT A CAKE FAIRLY

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Steinhaus [16] has written about the problem of dividing an object (such as a cake) among a finite number of people so that each is satisfied that he has received his fair share, though each may have a different opinion as to which parts of the cake are most valuable. In that article he presented, among other things, the affirmative solution of Banach-Knaster to this problem. Their solution stimulated us to ask whether it may not sometimes be possible to divide the cake so that each person feels that he receives strictly more than his share. We found that such divisions do exist whenever two or more of the participants value some part of the cake differently. It then seemed natural to try to formulate a definition of one division being better than another and to find out whether there exist best or optimal divisions.

In this paper we observe that positive solutions to these problems can be based on results in the literature, and relations, some old, some perhaps new, between these problems and other topics in the mathematical literature will be pointed out.

In Part I we restate some, but not all, of the ideas contained in the article of Steinhaus [16]. We also present other well-known problems involving the division or partitioning of one or more objects. It is intended that this part of the paper be a complete unit requiring little technical knowledge and that it can be read profitably by a reader whether or not he goes on to the second, and more technical, part of the paper.

In Part II the mathematical details are presented. The results are not new and center around some results of Lyapunov [13] and generalizations of these.

I. DIVISION PROBLEMS

We begin with the problem of dividing a cake between two people so that each is satisfied that he gets at least half the cake. Whether or not the two people agree as to what constitutes half the cake makes no difference as it is always possible to divide the cake in the desired manner.

A simple and well-known method of effecting such a division is for "one to cut, the other to choose." The one who cuts does not have a chance of receiving more than half the cake according to his measure unless he is willing to take the risk of receiving less than half. On the other hand, the one who chooses may have the opportunity to get more than half the cake according to his measure without incurring any such risk. Thus in a certain sense the procedure gives an advantage to the one who chooses. Nevertheless, it is fair in the sense that each person can ensure that he receives at least half the cake according to his own tastes independently of what the other does.

* This paper was prepared while one of its authors was a Regular Postdoctoral N.S.F. Fellow and the other was supported in part by the National Science Foundation under grant number G-10700.

Steinhaus [16] asked whether a fair procedure could be found for dividing a cake among n participants for $n>2$. He found a solution for $n=3$, and Banach and Knaster showed that the solution for $n=2$ can be extended in an elegant and simple way to arbitrary n.

Their solution to the problem is, in essence, as follows. A knife is slowly moved at constant speed parallel to itself over the top of the cake. At each instant the knife is poised so that it could cut a unique slice of the cake. As time goes by the potential slice increases monotonely from nothing until it becomes the entire cake. The first person to indicate satisfaction with the slice then determined by the position of the knife receives that slice and is eliminated from further distribution of the cake. (If two or more participants simultaneously indicate satisfaction with the slice, it is given to any one of them.) The process is repeated with the other $n-1$ participants and with what remains of the cake.

To show this is a fair method of distribution suppose that a participant (call him A) adopts the strategy of indicating satisfaction at that position of the knife where the piece cut off has a value of $1/n$th of the total value of the cake according to his measure. Then, independent of the strategies of the other participants, even allowing for coalitions and duplicity, it is assured that either (i) A is given the first piece of the cake, which is precisely $1/n$th of the cake according to his taste, or (ii) A becomes one of the $n-1$ participants who have a share in the remains of the cake, which is worth at least $(n-1)/n$ of the original cake according to A's own evaluation. It is easy to see (by induction on n) that A possesses a strategy which ultimately yields him at least $1/n$th of the original cake according to his own evaluation.

Of course, the solution of Banach and Knaster shows that there exists a division of the cake into n pieces such that the jth piece is worth at least $1/n$th of the cake according to the jth measure. But their solution is more than a mere existence theorem. In fact, it provides an important practical method for effecting such a division; moreover it is a method which does not require the services of an umpire or an expert to decide what the "true" value of each piece of the cake really is. Another important property of their solution is that the piece that each participant receives is an "elementary" set in a sense that is easily made precise.

The method described above is equally applicable for the division of any object provided only that (1) the value assigned by any participant to any part of the object equals the sum of the values of the subparts when the part is subdivided into any finite number of subparts; and (2) the value to each participant of the potential slice varies in a continuous fashion as the knife is moved over the object. The method is also applicable when the participants are not to share equally but to share in some preassigned way as long as each one has been assigned a rational share of the cake. Thus, suppose α_i is a nonnegative rational number with $\sum\alpha_i=1$ for $1\leqq i\leqq n$. Then the method above can be applied to effect a division in which the ith participant receives at least α_i of the total value of the object according to his evaluation. To see this, express each α_i as the ratio of two integers where the denominator is common to all i (so $\alpha_i = r_i/k$). Now make believe there are k participants, and let the real ith participant have a multiple personality and play the role of r_i of the fictitious participants. The method of division described above guarantees him $1/k$th of the cake in each of his roles. Therefore, he is assured of at least r_i/k of the cake according to his evaluation.

The constructive method for dividing a cake among n people described above gives each person a fair share according to his taste. However, it may result in a distribution for which one or more of the participants feels that someone else got more (or less) than his fair share. Steinhaus in [16] asserted that using methods similar to those of Stone and Tukey [17] it is possible to prove the existence of a partition of the cake into n sets so that each set has measure equal to $1/n$th of the total value of the cake to each participant (he actually asserted more, namely Corollary 1.1 below for the case $k=n$).

The problem considered above involves partitioning the cake into n sets and then evaluating each of n measures on each of the sets. We now describe another problem, a special case of which involves partitioning a set into k pieces and then evaluating each of n measures on each piece. The problem is called the "Problem of the Nile," and below we state a slightly modified version of Fisher's presentation of it [8, 9].

"Each year the Nile would flood, thereby irrigating or perhaps devastating parts of the agricultural land of a predynastic Egyptian village. The value of different portions of the land would depend upon the height of the flood. In question was the possibility of giving to each of the k residents a piece of land whose value would be $1/k$ of the total land value no matter what the height of the flood."

The problem as described above allows an infinite number of flood heights and in such a case, as shown by Feller [7], need not have a solution. Assuming that there are only a finite number, say n, of possible flood heights for the Nile, then Corollary 1.1 below (with $\alpha_j = 1/k$ for each j) shows the existence of a solution. That the problem has a solution under this hypothesis was first noted by Neyman [14]. Another contribution to this problem of the Nile is due to Tukey [19].

Closely related to the problem of the Nile is the "Problem of Similar Regions" of Neyman and Pearson [15]. Instead of partitioning a set, this problem involves for each α between 0 and 1, the existence of a single subset such that for each of n measures the ith measure of the subset is to the ith measure of the whole set as α is to 1. This problem, seemingly easier than the problem of the Nile, especially if α is rational, is actually equivalent to it as was first observed by Darmois [4].

The special case of the problem of similar regions in which the preassigned ratio is equal to $\frac{1}{2}$ we shall call the "bisection problem." It is known (and we shall actually see in Part II) that a solution of the bisection problem leads to a solution of the problem of the Nile so that they are, in fact, equivalent.

Other problems involving bisection have been considered, including various forms of the "Ham Sandwich Problem" [**17, 18**]. This stimulating problem of Ulam, according to Tucker [**18**], got its name because Steinhaus picturesquely formulated it as the problem of cutting a ham sandwich composed of ham, butter, and bread into two parts by one slice of a knife in such a way that each ingredient is halved.* The general problem involves simultaneously cutting in half each of n bodies in euclidean n-space by a hyperplane. For our purposes we shall consider the following slightly more general problem. We suppose we are given n finite measures in euclidean n-dimensional space such that each measure vanishes on any set whose volume is zero, and we seek to find a hyperplane bisecting the space with respect to each measure. Lemma 5.1 of Part II is related to this problem, and the proof of this lemma can be modified slightly to solve this form of the ham sandwich problem.

In Part II it will be shown that a solution to the ham sandwich problem (or Lemma 5.1) implies the solution of the general bisection problem of any set. These two problems are not equivalent, however, because the ham sandwich problem involves not merely bisection but bisection in a special way, namely by a hyperplane.

In Part II the standard proof of the ham sandwich problem (and Lemma 5.1) using the Borsuk-Ulam theorem [**3, 17, 18, 19**] is presented. This theorem asserts

If f is a continuous map of the surface of the sphere in n-dimensional space into $(n-1)$-dimensional space such that $f(-x)=-f(x)$ for every x then there is some point on the sphere mapped into the origin.

Thus, this topological theorem, which does not involve division or measure in its statement, provides a logical starting point for the solution of the problems discussed above, though, historically, Ulam's ham sandwich problem seems to have come first.

II. EXISTENCE THEOREMS

1. Introduction. Let $u=u_1, \cdots, u_n$ be an n-tuple of countably additive finite real-valued functions defined on a σ-algebra $\mathfrak{U}$ of subsets of a set U. (Though the special case in which the u_i are probability measures is of primary interest here, it is convenient to treat the general case.) With every ordered partition P of U into k measurable sets $A_1, \cdots, A_k$, $A_j \in \mathfrak{U}$ for $1 \leqq j \leqq k$, we associate the $n \times k$ matrix of real numbers $M(P)=(u_i(A_j))$. Our main objective is to show that the range R of the matrix-valued function M is compact. The study of the case $k=2$ is essentially the same as the study of the range of vector-valued measures $u_1(A), \cdots, u_n(A)$ as A ranges over $\mathfrak{U}$. This case has been

* Interesting further developments were obtained by Steinhaus. See Fund. Math., vol. 33, 1945, pp. 245–263, with an historical footnote p. 255.

treated by Lyapunov [10, 13] who showed that R is compact. It is noteworthy that in order to prove closure he had to observe, and prove, that if each u_i is nonatomic then R is convex. A simplified proof of Lyapunov's results has been given by Halmos [10], and interesting extensions of Lyapunov's theorems were obtained by Blackwell [1, 2] and by Dvoretsky, Wald and Wolfovitz [5, 6] and Karlin [12]. Our principal result is a special case of theirs and is indeed explicitly formulated and proven in [6]. Our proof that R is convex if each u_i is nonatomic is based on the measure-algebra isomorphism theorem of Halmos and von Neumann [11] and on the Borsuk-Ulam theorem. The main theorem is:

THEOREM 1. *If u is nonatomic, then R is a compact convex set of matrices.*

This result will be proved in sections 5 and 6. We observe, however, that it follows from Theorems 3 and 5 of [2]. That is, identifying $n\times k$ matrices with nk-dimensional euclidean space, let A be the closed subset of this euclidean space composed of the k points $p_1, \cdots, p_k$, where p_j is the matrix having 1's in the jth column and 0's elsewhere. Consider the nk measures v_{ij}, $1\leqq i\leqq n$, $1\leqq j\leqq k$, where $v_{ij}=u_i$; then our set R is exactly the range considered by Blackwell (because a measurable function f from U to this set A corresponds to a measurable partition $P=A_1, \cdots, A_k$ such that $f(A_j)=p_i$).

We draw some consequences of this result for the cake problem. Let α_j be a set of n nonnegative numbers such that $\sum\alpha_j=1$. We are interested in giving the ith person α_i of the cake in terms of the measure u_i. If $\alpha_i=0$, the ith person has no share in the cake, and the number of participants is less than n. Therefore, there is no loss of generality in assuming each $\alpha_j>0$. The following result shows that U can be partitioned so that each person believes the jth person receives α_j of U.

COROLLARY 1.1. *If each u_i is a nonatomic probability measure, then given k and $\alpha_1, \cdots, \alpha_k\geqq 0$ with $\sum\alpha_j=1$, there exists a partition $A_1, \cdots, A_k$ of U such that $u_i(A_j)=\alpha_j$ for all $i=1, \cdots, n$ and $j=1, \cdots, k$.*

Proof. Let P_j $(j=1, \cdots, k)$ be the partition in which $A_j=U$ and A_r is empty if $r\neq j$. Then $M(P_j)$ is the matrix having 1's in the jth column and 0's everywhere else. Theorem 1 implies that $\sum\alpha_jM(P_j)$ is in R. Hence, there exists a partition P such that the jth column of $M(P)$ equals the jth column of $\sum\alpha_jM(P_j)$ and therefore P has all its entries equal to α_j.

This corollary confirms the assertion of Steinhaus. It also yields an affirmative answer to the problem of the Nile provided that there are only a finite number of flood heights.

The next result shows that each person can be given strictly more than his share of U (*i.e.*, the jth person receives more than α_j of U) provided that there are at least two people with different measures. A different proof of the result, which does not use the convexity theorem, has been shown to us by Jacob Feldman.

COROLLARY 1.2. *Suppose that each u_i is a nonatomic probability measure and $u_j \neq u_k$ for some $j \neq k$. Let $\alpha_i > 0$ with $\sum \alpha_i = 1$. Then there exists a partition $A_1, \cdots, A_n$ such that $u_i(A_i) > \alpha_i$ for each i.*

Proof. Suppose, for example, that u_1 and u_2 are not identical. Then for some measurable A, $u_1(A) > u_2(A)$. Let B be the complement of A. Then clearly $u_2(B) > u_1(B)$. Without loss of generality (by symmetry) we can suppose that $u_1(A)/\alpha_1 \geqq u_2(B)/\alpha_2$. Let P_0 be the ordered partition determined by giving A to 1, B to 2 and nothing to i if $i > 2$. For each $i > 1$ let P_i be the partition obtained by giving all of U to i and nothing to any other index. For each n-tuple x_i with $x_i \geqq 0$ and $\sum x_i = 1$ it follows from Theorem 1 that there is a partition $P = (A_1, \cdots, A_n)$ such that

$$M(P) = x_1 M(P_0) + \sum_{i \geqq 2} x_i M(P_i).$$

Letting D denote the diagonal of $M(P)$ (so that the ith entry of D is $u_i(A_i)$) and letting D_i denote the diagonal of $M(P_i)$, we see that $D = x_1 D_0 + \sum_{i \geqq 2} x_i D_i$. We shall try to choose the x_i so that all the entries of D are in the same ratios as the α_i. Hence, we want to solve the equations

$$x_1 u_1(A) = \lambda \alpha_1, \qquad x_1 u_2(B) + x_2 = \lambda \alpha_2, \qquad x_i = \lambda \alpha_i \qquad \text{for } i > 2.$$

Solving for x_i, summing, and using the fact that $\sum \alpha_i = 1$, we find that if we choose

$$\lambda = \left(1 + \frac{\alpha_1}{u_1(A)} [1 - u_1(A) - u_2(B)]\right)^{-1},$$

we have a solution. Now $u_1(A) + u_2(B) > 1$ so that $1 - u_1(A) - u_2(B) < 0$. Since $\alpha_1 / u_1(A) < 1$, it follows that $\lambda > 1$. Therefore, choosing x_i to satisfy the above equations for this value of λ, we find that the ith entry of D is $\lambda \alpha_i > \alpha_i$ for all i, and this completes the proof.

The reader primarily interested in Lyapunov's theorem (or Theorem 1) will find little difficulty in skimming over Sections 2–4.

2. Preliminaries. We shall introduce hypotheses only when we have a need for them. We start with a set U, the object to be partitioned, and a finite sequence of finite real valued functions $u_1, \cdots, u_n$ each defined on the same nonempty collection $\mathfrak{U}$ of subsets of U. If $A \in \mathfrak{U}$, then $u_j(A)$ is the value or utility of the set A to the jth person. If P is the ordered partition $A_1, \cdots, A_n$ of U into sets $A_j \in U$ and the jth person is given the set A_j, then there are various criteria which might be applied to decide when one such partition is better than another. One of the simplest such criteria is to assign to the partition P the value $\sum u_j(A_j)$ and to define one partition to be better than another if it has a greater value. Interest now focuses on the existence of best (or maximal) partitions in this sense.

If there exists at least one partition and at most a finite number, it is trivial that a best one exists. Such is the case, for example, if U is a finite set and $\mathfrak{U}$ is the collection of all its subsets. When U is infinite, however, it is generally necessary to impose restrictions on the u_j. For example, if U is the set of positive integers, $\mathfrak{U}$ the collection of all its subsets, u_1 any countably additive probability measure on $\mathfrak{U}$ such that every integer has positive measure, and u_2 is a finitely additive function on $\mathfrak{U}$ vanishing on finite subsets and such that $u_2(U)=1$, then it is easy to see that no best partition exists. However, best partitions do exist if each utility function u_i is a countably additive measure as the following result shows.

THEOREM 2. *Let $\mathfrak{U}$ be a σ-algebra of subsets of U and let u_i, $1 \leqq i \leqq n$ be a countably additive real-valued measure on $\mathfrak{U}$. Then the supremum of $\sum u_i(A_i)$, $1 \leqq i \leqq n$ is attained as $P=A_1, \cdots, A_n$ ranges over ordered measurable partitions of U.*

Proof. Let v be any nonnegative finite-valued measure with respect to which each u_i is absolutely continuous (*e.g.*, v can be $\sum u_i$ if each u_i is nonnegative, or, more generally, $\sum |u_i|$, where $|u_i|$ is the total variation measure corresponding to u_i). Let f_i be a Radon-Nikodym derivative of u_i with respect to v (*i.e.*, f_i is a v-measurable function such that $u_i(A)=\int_A f_i dv$ for each $A \in \mathfrak{U}$). Let $f=\sup f_i$. We assert:

(i) For every P, $\sum u_i(A_i) \leqq \int f dv$.
(ii) There exists P such that $\sum u_i(A_i) = \int f dv$.

For (i) we observe that

$$\sum u_i(A_i) = \sum \int_{A_i} f_i dv \leqq \sum \int_{A_i} f dv = \int f dv.$$

For (ii) let A_i be the subset of U where $f_j < f$ for $j < i$ and $f_i = f$. Then A_i is measurable and clearly $A_1, \cdots, A_n$ is a partition of U. For this partition we have

$$\sum u_i(A_i) = \sum \int_{A_i} f_i dv = \sum \int_{A_i} f dv = \int f dv,$$

which completes the proof.

Let us return to the general case of any nonempty collection $\mathfrak{U}$ of subsets of a set U and real-valued functions $u_1, \cdots, u_n$ defined on $\mathfrak{U}$. We now discuss another notion of optimal partition somewhat more complicated than the above. The idea is to find a partition that maximizes the amount received by the person who gets the least, and, among such partitions, to find one that maximizes the amount received by the person who gets next to the least, etc. More precisely, for each partition $P=A_1, \cdots, A_n$ arrange the numbers $u_j(A_j)$ in nondecreasing order and call the resulting sequence

$$a_1(P) \leqq \cdots \leqq a_n(P).$$

P is called an *optimal* partition if for any partition P' either $a_i(P) = a_i(P')$ for all i, or if j is the smallest i such that $a_i(P) \neq a_i(P')$, then $a_j(P') < a_j(P)$ (*i.e.*, P is maximal in the ordering of partitions defined by lexicographic ordering of the sequences $a_1(P), \cdots, a_n(P)$). Thus if Q is the partition $B_1, \cdots, B_n$, then $\max_Q \min_i u_i(B_i) = a_1(P)$, and among all partitions Q such that $\min_j u_j(B_j) = a_1(P)$, then if $u_i B_i = a_1(P)$, $a_2(P) = \max_Q \min_{j \neq i} u_j(B_j)$, etc.

In a similar way if we were attempting to partition U according to the ratios α_i (*i.e.*, so that $u_i(A_i) \geqq \alpha_i$), where $\alpha_i > 0$ and $\sum \alpha_i = 1$, we let

$$a_1(P) \leqq \cdots \leqq a_n(P)$$

be the nondecreasing order of the numbers $u_i(A_i)/\alpha_i$, and then we order partitions lexicographically by these numbers. Thus optimal partitions in this sense would correspond to partitions P in which the smallest ratio $u_i(A_i)/\alpha_i$ was as large as possible, etc.

Clearly the compactness of the set of vectors $u_1(A_1), \cdots, u_n(A_n)$ implies the existence of optimal partitions. Thus attention is turned to this matter of compactness. The vector $u_1(A_1), \cdots, u_n(A_n)$ is the diagonal of the matrix $(u_i(A_j))$, where $1 \leqq i, j \leqq n$. This matrix represents the amount that each person believes each participant receives in the partition $A_1, \cdots, A_n$. The study of this set of matrices is, therefore, of interest, and, in particular, the compactness of this set implies the compactness of the set of diagonals. More generally, for each positive integer k, let $R = R(k)$ be the collection of all $n \times k$ matrices $(u_i(A_j))$ as $A_1, \cdots, A_k$ ranges over all partitions of U into k sets $A_j \in \mathcal{U}$. Our objective is to find conditions implying the compactness of R.

3. Compactness of R in certain special cases. To prove the compactness of R it would clearly suffice to find a topology on the set $\mathcal{P}$ of k-partitions $P = A_1, \cdots, A_k$ such that (i) $\mathcal{P}$ is compact, and (ii) if $M(P) = (u_i(A_j))$, then M is a continuous matrix-valued function on $\mathcal{P}$. Since $\mathcal{P}$ is a subset of $(\mathcal{U})^k$, the k-fold cartesian product of $\mathcal{U}$, it would suffice to find a topology on $\mathcal{U}$ such that (1) $\mathcal{U}$ is compact, (2) $\mathcal{P}$ is a closed subset of $(\mathcal{U})^k$, and (3) u is a continuous vector-valued function on $\mathcal{U}$.

When $\mathcal{U}$ is the collection of all subsets of U, there is a natural compact topology to put on $\mathcal{U}$. In fact, the set of all subsets of U is in 1-1 correspondence with the cartesian power over U of a 2-point space, and this has a compact topology under the product topology. In this topology a directed family A_j of subsets of U converges if and only if $\limsup A_j = \liminf A_j$ (where, as usual, $\limsup A_j = \bigcap_j \bigcup_{j'>j} A_{j'}$, $\liminf A_j = \bigcup_j \bigcap_{j'>j} A_{j'}$). Since the maps $A, B \rightarrow A \cup B$ and $A, B \rightarrow A \cap B$ are both jointly continuous in this topology, it follows that $\mathcal{P}$ is a closed subset of $(\mathcal{U})^k$. Therefore, this topology on $\mathcal{U}$ satisfies (1) and (2) above, and to verify (3) we need only show that u is continuous on $\mathcal{U}$. This is a reasonable condition to impose in the case U is countable, but if U is nondenumerable this is too restrictive a condition. A better one is to ask that whenever a *sequence* of sets A_p converges to a a set A, then $u(A_p)$ should converge to $u(A)$. In the event U is countable and $\mathfrak{U}$ is the collection of all its subsets this condition is equivalent to continuity because in this case $\mathfrak{U}$ is a compact metric space and so the topology is completely described by its convergent sequences. Therefore, we have proved:

THEOREM 3. *Let $u=u_1, \cdots, u_n$ be defined on the collection $\mathfrak{U}$ of all subsets of a countable set U with values in n-dimensional euclidean space. Suppose that whenever $A_p \to A$ in U then $u(A_p)\to u(A)$. Then the collection of matrices $(u_i(A_j))$, $1\leqq i\leqq n$, $1\leqq j\leqq k$, as $A_1, \cdots, A_k$ ranges over all partitions of U is a compact set.*

When U is uncountable and $\mathfrak{U}$ is a σ-algebra of subsets of U, we do not know if the above sequential condition on u is enough to imply the compactness of R. Indeed, this is unknown even if $\mathfrak{U}$ is the collection of all subsets of U. In fact we do not know whether there are any functions u other than the function identically zero defined on all subsets of U satisfying the condition $A_p \to A$ implies $u(A_p)\to u(A)$ and vanishing on countable A. If u is also assumed to be finitely additive (and, therefore, countably additive) Ulam [20] has shown the nonexistence of a nontrivial u for sets U of appropriate cardinality.

4. The purely atomic case. Henceforth we shall suppose that u is a countably additive finite real-valued measure defined on a σ-algebra $\mathfrak{U}$ of subsets of U. A measurable set A is said to be an *atom* for u if $u(A)\neq 0$ and for every measurable set B either $u(A\cap B)=0$ or $u(A-B)=0$. Clearly if A is an atom for u then for each i it is also an atom for u_i if $u_i(A)\neq 0$. It is worth noting that the converse does not hold. For example, let U consist of two distinct points a_1 and a_2, let $\mathfrak{U}$ consist of the four subsets of U, and let

$$u_i(a_j) = \begin{cases} 1 & \text{if } i=j, \\ 0 & \text{if } i\neq j. \end{cases}$$

Then U is an atom for both u_1 and u_2, but it is not an atom for u.

A measure u is said to be *purely atomic* if there exists a disjoint collection K of atoms for u with the property that for each measurable set B, $u(B) = \sum u(B\cap A)$, $A\in K$. u is said to be *nonatomic* provided that there are no atoms for u. We observed above that an atom for each u_i need not be an atom for u; however, it is true that u is purely atomic if and only if each u_i is. We also have the following result:

LEMMA 4.1. *u is nonatomic if and only if each u_i is nonatomic.*

Proof. We saw above that an atom for u is an atom for at least one u_i. Therefore, if all the u_i are nonatomic, so is u.

To prove the converse we prove by induction that if A is an atom for the vector measure $v_k=u_1, \cdots, u_k$, then there exists an atom for the vector measure $v_{k+1}=u_1, \cdots, u_{k+1}$. A is the union of A' and A'', where u_{k+1} is nonnegative on all subsets of A' and nonpositive on all subsets of A''. Either A' or A'' is again an atom for v_k. Without loss of generality, therefore, we can assume that we have an atom, say A, for v_k such that u_{k+1} is nonnegative on all subsets of A.

Consider the collection of subsets B of A which are atoms for v_k. This collection is easily seen to be closed under finite intersections, and therefore under countable intersections. Hence, there exists such a subset B_0 such that $u_{k+1}(B_0) \leqq u_{k+1}(B)$ for all such B. An easy verification shows that B_0 is an atom for v_{k+1} so the induction is complete, and the lemma is proved.

Given a measure u let K be a maximal disjoint collection of atoms for u (which exists by Zorn's lemma). It is trivial that K is at most denumerable so its union U_1 is measurable. Clearly u restricted to U_1 is purely atomic and restricted to $U-U_1$ is nonatomic. Hence, any measure u splits into a purely atomic part and a nonatomic part whose sum equals u.

If u is purely atomic, it is easy to verify that it is isomorphic to a measure defined on the set of all subsets of some countable set. Therefore, Theorem **3** can be applied and we obtain:

THEOREM 4. *Let u be a purely atomic n-dimensional vector measure. Then R, the range of the $n \times k$ matrix valued function of k-partitions, is compact.*

This result disposes of the purely atomic case, and we now pass on to the consideration of the more interesting nonatomic case.

5. Lyapunov's convexity theorem. The proof of Lyapunov's theorem presented here uses the following two results. (For a proof of the first see [**3**], [**18**], p. 293 while the second is proved in [**11**], Theorem C, p. 173.)

Borsuk-Ulam Theorem. Let f be a continuous mapping of the n-dimensional sphere S^n into n-dimensional euclidean space such that $f(x)+f(-x)=0$ for every $x \in S^n$. Then there exists $x \in S^n$ such that $f(x)=0$.

Halmos-Von Neumann Isomorphism Theorem. Any two separable nonatomic measure algebras of countably additive probability spaces are isomorphic.

The convexity theorem we are after asserts that the range R of our $n \times k$ matrix-valued function of partitions is convex if u is nonatomic. This is a corollary of Lyapunov's theorem which asserts that the range of a nonatomic vector-valued measure is convex. We present a sequence of lemmas leading to these results.

LEMMA 5.1. *Let $\mathcal{S}$ be the σ-algebra of Borel subsets of S^n, let w be the usual rotation invariant measure on $\mathcal{S}$, and let $w_1, \cdots, w_n$ be n countably additive real-valued measures on $\mathcal{S}$ each absolutely continuous with respect to w. Then there exists a closed hemisphere E of S^n such that for all i, $w_i(E)=\frac{1}{2}w_i(S^n)$.*

Proof. For each $x \in S^n$ let $E(x)$ be the closed hemisphere of all $y \in S^n$ whose inner product with x is nonnegative. Then $E(x) \cup E(-x) = S^n$ and

$w(E(x) \cap E(-x))=0$. If x and x' are close points of S^n, then the symmetric difference of $E(x)$ and $E(x')$ has small w-measure. Since each w_i is absolutely continuous with respect to w, it also has small w_i measure. Therefore, for each i, the function $w_i(E(x))$ is a continuous function on S^n. Define f on S^n by

$$f(x) = (w_1(E(x)) - \tfrac{1}{2}w_1(S^n), \cdots, w_n(E(x)) - \tfrac{1}{2}w_n(S^n)).$$

Then f is a continuous map of S^n into euclidean n-space such that $f(x)+f(-x)=0$ for all $x \in S^n$. The Borsuk-Ulam theorem implies the existence of $x \in S^n$ such that $f(x)=0$. Then $E(x)$ satisfies the condition that $w_i(E(x))=\frac{1}{2}w_i(S^n)$ for all i.

COROLLARY. *Let $E_1, \cdots, E_n$ be measurable subsets of S^n. Then there exists a hemisphere that contains half of each E_i.*

We observe in passing that the techniques used in [17] also yield the following slightly generalized version of the ham sandwich theorem: *Given any n countably additive finite real-valued measures defined on the Borel subsets of n-dimensional euclidean space that are absolutely continuous with respect to Lebesgue measure, there exists a hyperplane that bisects each measure.*

Using the Halmos-Von Neumann isomorphism theorem we now show that Lemma 5.1 implies the solution of the bisection problem in any space.

The structure of this section is represented schematically by the following diagram showing the implications to be established and the number of the lemma in which the implication in question is proved.

Borsuk-Ulam Theorem $\overset{(5.1)}{\Longrightarrow}$ Ham Sandwich Theorem

$\overset{(5.2)}{\Longrightarrow}$ General Bisection Theorem

$\overset{(5.3)}{\Longrightarrow}$ Existence of monotone one-parameter family of subsets

$\overset{(\text{Theorem } 5)}{\Longrightarrow}$ Lyapunov Convexity Theorem $\overset{(5.4)}{\Longrightarrow}$ Matrix convexity.

LEMMA 5.2. *Let u be a nonatomic countably additive vector-valued measure defined on a σ-algebra $\mathfrak{U}$ of subsets of U. Then there exists a set A of $\mathfrak{U}$ such that $u(A)=\frac{1}{2}u(U)$.*

Proof. Let $v=\sum|u_i|$, where $|u_i|$ is the total variation of the ith component of u. Then by (4.1), v is a nonatomic positive measure. It can be verified that either the measure ring of v is separable or else it contains a nonatomic separable subalgebra. Let v' denote the restriction of v to such a separable subalgebra. By the Halmos-Von Neumann theorem the measure ring of v' is isomorphic to the measure ring $\mathcal{S}$ on S^n so that v' corresponds to the measure w. Each u_i has a restriction u_i' to the subalgebra which corresponds to a measure w_i on $\mathcal{S}$, and since u_i is absolutely continuous with respect to v so is w_i with respect to w. By (5.1) there is a closed hemisphere E of S^n such that $w_i(E)=\frac{1}{2}w_i(S^n)$ for each i. Let A

be the element of $\mathfrak{U}$ corresponding to E (under the isomorphism of the separable subalgebra with $\mathcal{S}$). Then $u_i(A)=\frac{1}{2}u_i(U)$ for all i, completing the proof.

The next result shows that the solution of the bisection problem can be used to find a one-parameter monotone family of subsets $A(t)$ such that $u(A(t)) = tu(U)$.

LEMMA 5.3. *Let u be a nonatomic vector measure as in* (5.2). *There exists a function $t \rightarrow A(t)$ mapping the closed unit interval $[0, 1]$ into $\mathfrak{U}$ such that*:

(1) $u(A(t)) = tu(U)$, (2) $t_1 < t_2 \Rightarrow A(t_1) \subset A(t_2)$.

Proof. Let Q be the collection of functions $t \rightarrow A(t)$ defined on subsets of $[0, 1]$ satisfying (1) and (2) where defined. We define a partial ordering in Q by saying one function is larger than another if its domain of definition is larger and its restriction to the smaller domain equals the smaller function. Zorn's lemma can be applied to Q to prove the existence of a maximal function A. We need only show A is defined on all of $[0, 1]$. Let D be the domain of A. Then if t_i is a monotone sequence in D converging from below (or from above) to a t not in D, we can extend A to $D \cup \{t\}$ by defining $A(t)=\bigcup A(t_i)$ (or $A(t)=\bigcap A(t_i)$). This contradicts the maximality of A. Therefore, monotone sequences in D converge in D so that D is closed.

Now suppose D is not all of $[0, 1]$. Since 0 and 1 are clearly in D (because if not we could extend A by defining $A(0)=\emptyset$ and $A(1)=U$) and D is closed, there exist $t_1, t_2 \in D$ with $t_1 < t_2$ such that no point of the open interval (t_1, t_2) is in D. By (2) we have $A(t_1) \subset A(t_2)$. The restriction of u to the measurable subsets of $A(t_2)-A(t_1)$ satisfies the hypothesis of (5.2) so there exists a measurable subset B of $A(t_2)-A(t_1)$ such that $u(B)=\frac{1}{2}u(A(t_2)-A(t_1))$. We can extend A to $D \cup \{\frac{1}{2}t_1+\frac{1}{2}t_2\}$ by defining $A(\frac{1}{2}t_1+\frac{1}{2}t_2)=B \cup A(t_1)$. This contradicts the maximality and therefore $D=[0, 1]$.

It should be noticed that Lemma 5.3 is equivalent to a generalization of an interesting theorem discovered by Neyman [**14**]. *Given any n nonatomic countably additive finite real-valued measures u_i, $i=1, \cdots, n$ defined on a σ-algebra $\mathfrak{U}$ of subsets of a set U, there exists a measurable function f such that $u_i[f^{-1}(-\infty, t)] = tu_i(U)$.*

We now show that (5.3) yields a proof of Lyapunov's convexity theorem.

THEOREM 5 (Lyapunov). *The range of a nonatomic vector-valued measure is convex.*

Proof. Let A_1, A_2 be two elements of $\mathfrak{U}$ and let $t_1, t_2 \geqq 0$ with $t_1+t_2=1$. Consider the $2n$-dimensional vector-valued measure m defined on $\mathfrak{U}$ by $m(B) = (u(B \cap A_1), u(B \cap A_2))$. Lemma 5.3 implies the existence of a set B such that $m(B)=t_1 m(U)$ or, equivalently,

$$u(B \cap A_1) = t_1 u(A_1), \qquad u(B \cap A_2) = t_1 u(A_2).$$

Therefore, letting B^C = complement of B in U, we have

$$u(B^C \cap A_2) = u(A_2) - u(B \cap A_2) = (1 - t_1)u(A_2) = t_2u(A_2).$$

Let $A = (B \cap A_1) \cup (B^C \cap A_2)$. Then

$$u(A) = u(B \cap A_1) + u(B^C \cap A_2) = t_1u(A_1) + t_2u(A_2),$$

completing the proof.

COROLLARY 5.4. *For any positive integer k if u is a nonatomic vector-valued measure then the range $R(k)$ of the $n \times k$ matrix-valued function M of k-partitions $P = A_1, \cdots, A_k$ defined by $M(P) = (u_i(A_j))$ is convex.*

Proof. Let $A_1, \cdots, A_k$ and $A_1', \cdots, A_k'$ be two partitions of U and let $0 < t < 1$. Let m be the $2n \times k$ vector-valued measure determined by the real-valued measures $B \rightarrow (u_i(B \cap A_j), u_i(B \cap A_j'))$ as B ranges over $\mathfrak{U}$. By Theorem 5 there exists a set B such that $u_i(B \cap A_j) = tu_i(A_j)$ and $u_i(B \cap A_j') = tu_i(A_j')$. Therefore,

$$u_i(B \cap A_j) = tu_i(A_j), \qquad u_i(B^C \cap A_j') = (1 - t)u_i(A_j').$$

It follows that if we set $E_j = (B \cap A_j) \cup (B^C \cap A_j')$, then

$$u_i(E_j) = tu_i(A_j) + (1 - t)u_i(A_j').$$

The simple verification that $E_1, \cdots, E_k$ forms a partition of U completes the proof. (The device of stringing together a number of vector-valued measures is borrowed from Blackwell [**2**].)

6. Compactness of R. Lyapunov proved not only that the range of a vector valued measure is convex but also that it is compact. Though the convexity of R was shown to be a simple corollary to Lyapunov's convexity theorem via Blackwell's device, the compactness of R does not follow easily, if at all, from Lyapunov's compactness theorem. Therefore a proof that R is compact will be given here that has the same general pattern as Halmos' presentation of Lyapunov's compactness theorem [**10**].

Consider first the nonatomic case. We know that R is bounded in the nk-dimensional euclidean space E, and convex by (5.4). For any set R to be closed it of course suffices that $\overline{R} - R$ be empty (where $\overline{R}$ denotes the closure of R). For any convex set R, and any $y \in \overline{R} - R$, there is a supporting hyperplane H of $\overline{R}$ not containing all of R with $y \in H$. To see this let E' be the affine variety generated by R. A standard separation theorem for convex sets implies the existence of a hyperplane of support H' of $\overline{R}$ in E' that contains y. Let H be any hyperplane of E whose intersection with E' is H'. Then H is the desired hyperplane, because $y \in H$, and, since H' is a proper subvariety of E', there is an $r \in R$ not in H'. Certainly $r \in E'$. But $H \cap E' = H'$, so $r \notin H$.

Returning to the general program, it now follows that to show $\overline{R} - R$ is empty, it suffices to show that if H is a supporting hyperplane of $\overline{R}$ not containing all of R, then $\overline{R} \cap H \subset R$. Therefore the proof that R is compact is reduced to establishing the following two propositions

$$\overline{R} \cap H \subset \overline{R \cap H}, \tag{6.1}$$

$$R \cap H \text{ is closed.} \tag{6.2}$$

The hyperplane H is associated with a linear function t and a constant c such that

(i) $t(x) \leqq c$ for all $x \in R$, (ii) $t(x) = c$ for all $x \in H$,

(iii) for all $\epsilon > 0$ there exists $x \in R$ such that $t(x) > c - \epsilon$.

Consider the following proposition which implies (6.1)

(6.3) *Given $\epsilon > 0$ there exists $\delta > 0$ such that for $x \in R$ with $t(x) > c - \delta$, there exists $x^* \in R$ such that $t(x^*) = c$ and $|x_{ij} - x^*_{ij}| < \epsilon$ for $1 \leqq i \leqq n$, $1 \leqq j \leqq k$.*

Proof that (6.3)$\Rightarrow$(6.1). Let $y \in \overline{R} \cap H$. To prove (6.1) we show that every 2ϵ-neighborhood of y meets $R \cap H$. Choose $\delta > 0$ according to (6.3). The set of x such that $t(x) > c - \delta$ is open and contains y. Since $y \in \overline{R}$, the intersection of this set with the ϵ-neighborhood of y meets R, so there is $x \in R$ such that $t(x) > c - \delta$ and $|y_{ij} - x_{ij}| < \epsilon$ for all i, j. Applying (6.3) we obtain a point $x^* \in R \cap H$ such that $|x_{ij} - x^*_{ij}| < \epsilon$ for all i, j. Then $|y_{ij} - x^*_{ij}| < 2\epsilon$ for all i, j, and the proof is complete.

We now concentrate on the proof of assertion (6.3) The linear function t is associated to an array t_{ij} of real numbers such that $t(x) = \sum t_{ij} x_{ij}$, $1 \leqq i \leqq n$, $1 \leqq j \leqq k$. Let $u'_j = \sum_i t_{ij} u_j$.

LEMMA 6.4. *The number c is the least upper bound of $\sum u'_j(A_j)$ as $A_1, \cdots, A_k$ ranges over all measurable partitions of U into k sets.*

Proof. If $x = M(P)$, where $P = A_1, \cdots, A_k$, then $t(x) = \sum_{i,j} t_{ij} u_i(A_j) = \sum_j u'_j(A_j)$. Since $c = \sup_{x \in R} t(x)$, the result follows.

Let $v = \sum |u_i|$ where $|u_i|$ is the total variation measure corresponding to u_i. Let f_j be a Radon-Nikodym derivative of u'_j with respect to v (which exists because u'_j is obviously absolutely continuous with respect to v). Let f be the supremum of the k functions $f_1, \cdots, f_k$ and σ an arbitrary nonempty subset of the k integers $1, \cdots, k$. Define F_σ to be the set of $z \in U$ such that $f_i(z) = f(z)$ for $i \in \sigma$ and $f_j(z) < f(z)$ for $j \notin \sigma$. It is clear that each F_σ is a measurable set, that any two are disjoint, and that their union is U; hence, they form a measurable partition of U. Furthermore, if i and i' are in σ, then the restriction of u'_i to F_σ is identical with the restriction of $u'_{i'}$ to F_σ, and we shall denote this restriction by u_σ.

LEMMA 6.5. $\sum u_\sigma(F_\sigma) = c$.

Proof. Given any partition $A_1, \cdots, A_k$ of U we have

$$(6.6) \qquad \sum_j u_j'(A_j) = \sum_{j,\sigma} u_j'(F_\sigma \cap A_j) \leqq \sum_{j,\sigma} u_\sigma(F_\sigma \cap A_j) = \sum_\sigma u_\sigma(F_\sigma).$$

This shows that $\sum u_\sigma(F_\sigma)$ is an upper bound for $\sum_j u_j'(A_j)$. If, however, we define a partition $A_1, \cdots, A_k$ by the condition that A_j is the union of those F_σ for which j is the smallest integer in σ, then we see that the inequality in (6.6) above is an equality and $\sum u_\sigma(F_\sigma) = \sum u_j'(A_j)$ for this partition. Therefore, $\sum u_\sigma(F_\sigma)$ is the least upper bound of the numbers $\sum u_j'(A_j)$ so, by (6.4), equals c.

LEMMA 6.7. *Suppose $j \notin \sigma$ and that A is a measurable subset of F_σ such that $v(A) > 0$. Then $u_\sigma(A) > u_j'(A)$.*

Proof. If $i \in \sigma$, we have

$$u_\sigma(A) - u_j'(A) = \int_A f_i dv - \int_A f_j dv = \int_A (f_i - f_j) dv.$$

But $f_i - f_j > 0$ on F_σ and hence on A. Since $v(A) > 0$ and v is countably additive, the right-hand side of the above equation is positive and the proof is complete.

LEMMA 6.8. *$\sum u_j'(A_j) = c$ if and only if for all σ and all $j \notin \sigma$, $v(F_\sigma \cap A_j) = 0$.*

Proof. $\sum u_j'(A_j) = c$ if and only if equality holds in (6.6). This is equivalent to

$$u_j'(F_\sigma \cap A_j) = u_\sigma(F_\sigma \cap A_j)$$

for all j, σ. The latter holds automatically if $j \in \sigma$, and for $j \notin \sigma$ is equivalent to $v(F_\sigma \cap A_j) = 0$, by (6.7), which completes the proof.

LEMMA 6.9. *Given $\epsilon > 0$ there exists $\delta > 0$ such that if $A_1, \cdots, A_k$ is a measurable partition with $\sum u_j'(A_j) > c - \delta$ then there exists a partition $B_1, \cdots, B_k$ such that:*

$$(1) \qquad \sum u_j'(B_j) = c, \qquad (2) \qquad v((A_j - B_j) \cup (B_j - A_j)) < \epsilon \text{ for all } j.$$

Proof. Lemma 6.7 shows that if $j \notin \sigma$ then, on F_σ, v is absolutely continuous with respect to $u_\sigma - u_j'$. Therefore, given $\epsilon > 0$ there exists δ such that for all pairs j, σ with $j \notin \sigma$, if $A \subset F_\sigma$ and $(u_\sigma - u_j')(A) < \delta$, then $v(A) < \epsilon/\bar{k}$ (where $\bar{k}$ is the number of pairs (j, σ) with $j \notin \sigma$). We show this δ satisfies the conditions of the lemma.

Let $A_1, \cdots, A_k$ be a partition with $\sum u_j'(A_j) > c - \delta$. Consider the partition $A_j \cap F_\sigma$ for all j, σ. We have

$$\sum_{j,\sigma} u_j'(A_j \cap F_\sigma) = \sum_j u_j'(A_j) > c - \delta = \sum_{j,\sigma} u_\sigma(A_j \cap F_\sigma) - \delta.$$

Therefore, $\sum_{j,\sigma} [u_\sigma(A_j \cap F_\sigma) - u_j'(A_j \cap F_\sigma)] < \delta$. Since each term of the sum is nonnegative, it follows that $u_\sigma(A_j \cap F_\sigma) - u_j'(A_j \cap F_\sigma) < \delta$ for all j, σ. Therefore, if $j \notin \sigma$ it follows from the choice of δ that $v(A_j \cap F_\sigma) < \epsilon/\bar{k}$.

For each $j \notin \sigma$ let $i(j, \sigma)$ be the smallest integer in σ. We define a partition $B_1, \cdots, B_k$ by

$$B_i = \bigcup\{A_i \cap F_\sigma \mid i \in \sigma\} \cup \bigcup\{A_j \cap F_\sigma \mid j \notin \sigma \text{ and } i = i(j, \sigma)\}.$$

Then $B_i \subset \bigcup_{i\in\sigma} F_\sigma$ so if $i \notin \sigma$, $B_i \cap F_\sigma$ is empty, and by (6.8), $\sum u_j^i(B_j) = c$, so the partition $B_1, \cdots, B_k$ satisfies (1). To see that it satisfies (2) we have

$$(A_i - B_i) \cup (B_i - A_i) \subset \bigcup_{j\notin\sigma} A_j \cap F_\sigma,$$

and therefore

$$v((A_i - B_i) \cup (B_i - A_i)) \leqq \sum_{j\notin\sigma} v(A_j \cap F_\sigma) < \bar{k}\cdot\epsilon/\bar{k} = \epsilon,$$

and this completes the proof.

Clearly (6.3) is an immediate consequence of (6.9) so it only remains to verify (6.2) to prove the compactness of R. The proof of compactness proceeds by induction on d = the dimension of R. If $d=0$, R is a single point and there is nothing to prove. Now we assume inductively that for all n, k whenever we have a nonatomic n-dimensional vector-valued measure on a measure space such that the dimension of the $n\times k$ matrix-valued function of k-partitions has dimension $<d$ then it is compact. Assume dim $R=d$. We show the inductive assumption implies that $R\cap H$ is closed for every supporting plane H not containing R (*i.e.*, it implies (6.2)), and then, having (6.1), we obtain the compactness of R, and the induction is complete.

It only remains, therefore, to prove $R\cap H$ is closed using the inductive hypothesis. Note that $\dim(R\cap H)$ is necessarily less than d because H does not contain R. We use the same notation as earlier in this section.

For each σ let R_σ be the set of all $n\times k$ matrices of the form $u_i(C_j)$, where $C_1, \cdots, C_k$ is a partition of F_σ such that $v(C_j)=0$ if $j\notin\sigma$. Given such a partition $C_j(\sigma)$ of F_σ for every σ, we define a partition $A_1, \cdots, A_k$ of U by setting $A_j = \bigcup_\sigma C_j(\sigma)$. Then (6.8) shows that $(u_i(A_j))$ is a matrix in $R\cap H$. Conversely, if $(u_i(A_j))$ is a matrix of $R\cap H$, (6.8) shows that for each σ, $A_j\cap F_\sigma = C_j(\sigma)$ is a partition of F_σ into sets $C_1(\sigma), \cdots, C_k(\sigma)$ such that $v(C_j(\sigma))=0$ if $j\notin\sigma$. The relation between the partition A_j and the partitions $C_j(\sigma)$ is such that $(u_i(A_j)) = \sum_\sigma (u_i(C_j(\sigma)))$. Therefore, $R\cap H = \sum_\sigma R_\sigma$. Since $\sum_\sigma R_\sigma$ lies in $R\cap H$, which has dimension $<d$, each R_σ has dimension less than d. By the inductive assumption each R_σ is compact (if $k(\sigma)$ is the number of elements of σ, then R_σ is clearly homeomorphic to the range of the $n\times k(\sigma)$ matrix-valued function defined on F_σ by partitions into $k(\sigma)$ sets and by the measures u_i). Therefore, $R\cap H$ is compact, and this completes the proof of compactness in the nonatomic case. This together with Theorem 5 completes the proof of Theorem 1.

Theorem 4 asserted the compactness of R in the purely atomic case. Combining this with the result just proved and using the decomposition of an arbitrary measure into its purely atomic and nonatomic parts as described in Section 4 we have the following.

THEOREM 6. *The range R of the $n\times k$ matrix-valued function M of partitions $P=A_1, \cdots, A_k$ defined by $M(P)=(u_i(A_j))$ is compact.*

Returning to the concept of optimal partition introduced in Section 2 we have the following.

COROLLARY 6.10. *There exist optimal partitions.*

We conclude with one further notion. An optimal partition $A_1, \cdots, A_n$ will be called *equitable* if $u_i(A_i) = u_j(A_j)$ for all i and j. In the event that the measures u_j are not only nonatomic but also each is absolutely continuous with respect to every other, then every optimal partition is equitable. Even in the nonatomic case the converse does not hold.

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